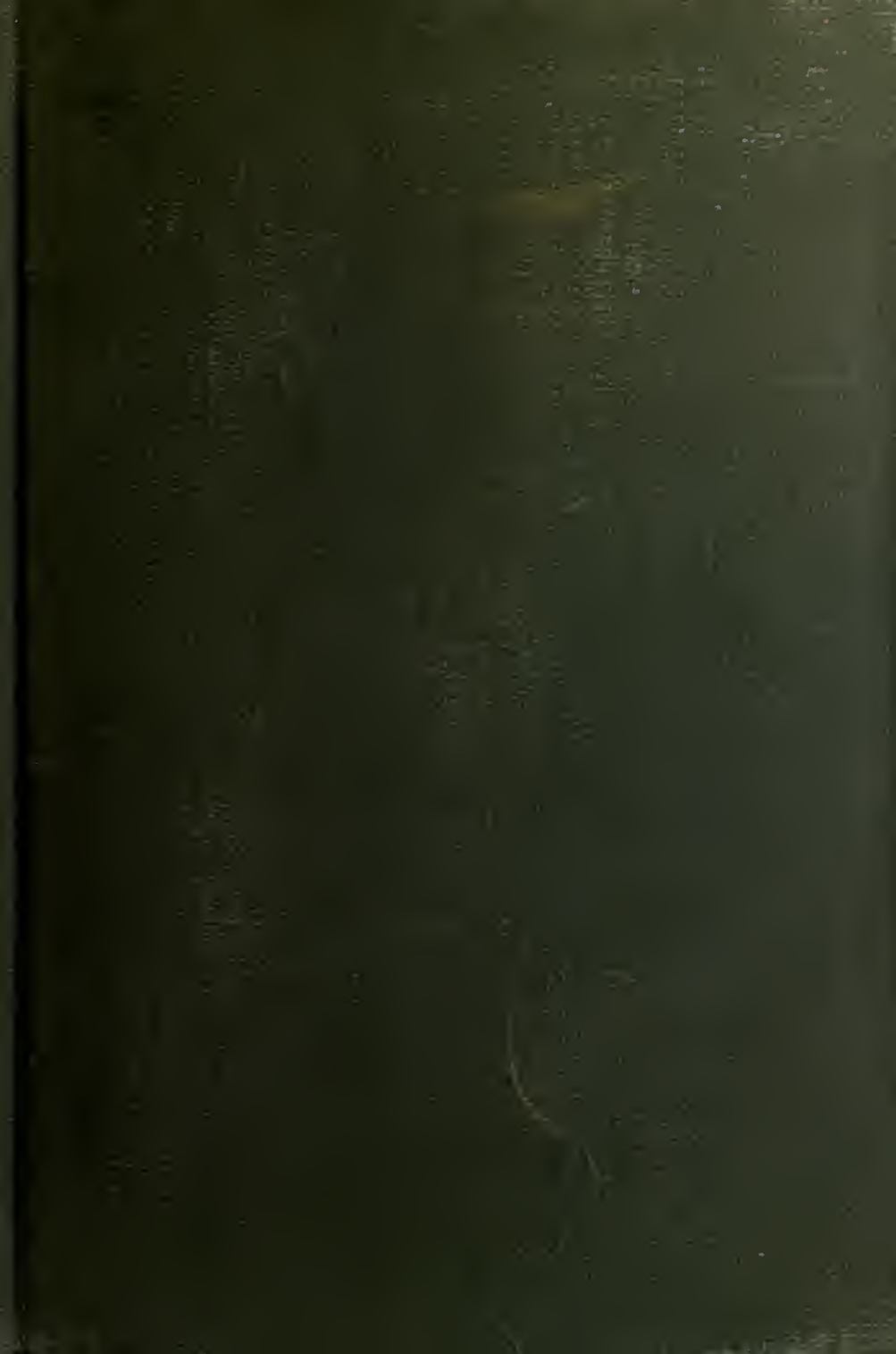

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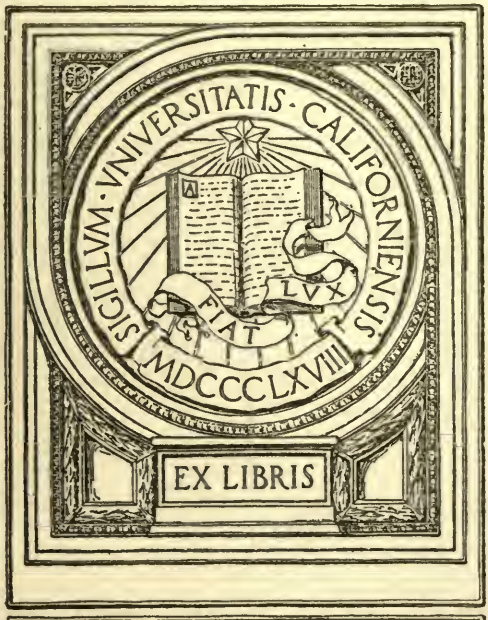

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A TREATISE

ON THE

INTEGRAL CALCULUS

FOUNDED ON THE

METHOD OF RATES

BY

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PREFACE.

THE present volume is an enlargement and an extension of my *Elementary Treatise on the Integral Calculus*, of which a revised edition was published in 1898. The enlargement consists chiefly in a fuller treatment of formulæ of reduction, and of double and triple integrals in connection with their geometrical representation by cylindrical volumes and solids of variable density respectively. The extension, which constitutes nearly half of the volume, consists of Chapter IV on Mean Values and Probabilities, Chapter V on Definite Integrals including the Eulerian Integrals, Fourier's Series, etc., and Chapter VI on Functions of the Complex Variable.

The book forms a companion volume to my *Treatise on the Differential Calculus, founded on the Method of Rates* (John Wiley & Sons, 1904) to which belong the references made in the text.

The Integral Calculus may be said to consist of two distinct parts: the first concerns the reduction of integral expressions to previously known functional forms; the second concerns the mode of expressing a required magnitude in the integral form for subsequent reduction when possible. While the fluxional foundation does not affect the treatment of the former (which is essentially an inverse process, depending upon a body of rules established in the Differential Calculus); in the latter, it leads us to regard the required magnitude as a particular value of a

varying magnitude, or fluent, of which the fluxion or rate of growth is to be ascertained.

In some of the simpler applications, the differential, or *hypothetical increment*, which, in the treatment of the subject here followed, measures this rate of growth, admits of construction, *e.g.* in Arts. 103 and 116. In general, however, it is necessary to regard the magnitude in question as the limit of a sum. The identification of this limit with the definite integral is effected in Art. 97 by the aid of the area which has already been shown to represent the integral as originally defined. Accordingly, the integral expression for the fluent magnitude is for the most part found in the usual manner by means of its element. But, while the idea of the limit is of course essential, the rate method presents the advantage that the integral is defined beforehand, so that the student realizes that it is the sum and not the integral which has an approximate character; in other words, the idea of approximation is not carried into the definition of the integral.

The direct representation of the fluent integral as the ordinate of a graph, or curve determined directly by the integral expression itself, flows naturally from the fluxional point of view, and is developed at some length in Arts. 85–91.

Somewhat more than the usual amount of space has been devoted to methods of integration and their classification in such manner that the form of the expression to be integrated shall suggest the best method of attack. A careful consideration of this matter has led me to lay particular stress upon trigonometric substitutions.

The subject of Mean Values of continuous variables forms one of the most useful and characteristic applications of the Integral Calculus. It seems therefore, next to the strictly geometrical applications, to be entitled to ample treatment, especially as it includes the important subjects of Centres of Gravity and Radii

of Gyration, of which the treatment is both simplified and shortened by reference to the underlying principles of Mean Values.

In the examples following Section XXI will be found a considerable collection of discontinuous multiple-angle series, and in the text, Arts. 324–327, an interesting illustration of the relation of such series to the ordinary multiple-angle series, and their connection with the occurrence of divergency.

To the section on the Γ-function is appended a logarithmic table of this function, and also a table of the values of s_n by means of which many interesting numerical computations may be made.

As in the companion volume, the sections are followed by large collections of examples with their answers. Many of the results being properties of well-known curves are rendered accessible by references inserted in the index.

W. Woolsey Johnson.

August, 1907.

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THE

INTEGRAL CALCULUS.

CHAPTER I.

Elementary Methods of Integration.

I.

Integrals.

1. In an important class of problems, the required quantities are magnitudes generated in given intervals of time with rates given in terms of the time t; or else, being assumed to be so generated concurrently with some other independent variable, have rates expressible in terms of this independent variable and its rate.

For example, the velocity of a freely falling body is known to be expressed by the equation

$$v = gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which t is the number of seconds which have elapsed since the instant of rest, and g is a constant which has been determined experimentally. If s denotes the distance of the body

at the time t, from a fixed origin taken on the line of motion, v is the rate of s; that is,

$$v = \frac{ds}{dt};$$

hence equation (1) is equivalent to

$$ds = gt\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which expresses the differential of s in terms of t and dt. Now it is obvious that $\frac{1}{2}gt^2$ is a function of t having a differential equal to the value of ds in equation (2); and, moreover, since two functions which have the same differential (and hence the same rate) can differ only by a constant, the most general expression for s is

$$s = \tfrac{1}{2}gt^2 + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which C denotes an undetermined constant.

2. A variable thus determined from its rate or differential is called an *integral*, and is denoted by prefixing to the given differential expression the symbol $\int$, which is called the integral sign.* Thus, from equation (2) we have

$$s = \int gt\,dt,$$

which therefore expresses that s is a variable whose differential is $gtdt$; and we have shown that

$$\int gt\,dt = \tfrac{1}{2}gt^2 + C.$$

The constant C is called the *constant of integration;* its occurrence in equation (3) is explained by the fact that we have not determined the origin from which s is to be measured.

* The origin of this symbol, which is a modification of the long s, will be explained hereafter. See Art. 100.

If we take this origin at the point occupied by the body when at rest, we shall have $s = 0$ when $t = 0$, and therefore from equation (3) $C = 0$; whence the equation becomes $s = \frac{1}{2}gt^2$.

The Differential of a Curvilinear Area.

3. The area included between a curve, whose equation is given, the axis of x and two ordinates affords an instance of the second case mentioned in the first paragraph of Art. 1; namely, that in which the rate of the generated quantity, although not given in terms of t, can be readily expressed by means of the assumed rate of some other independent variable.

Let BPD in Fig. 1 be the curve whose equation is supposed to be given in the form

$$y = f(x).$$

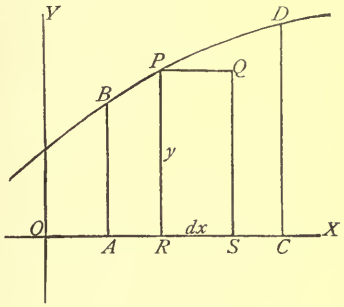

FIG. 1.

Supposing the variable ordinate PR to move from the position AB to the position CD, the required area $ABDC$ is the final value of the variable area $ABPR$, denoted by A, which is generated by the motion of the ordinate. The rate at which the area A is generated can be expressed in terms of the rate of the independent variable x. The required and the assumed rates are denoted, respectively, by $\frac{dA}{dt}$ and $\frac{dx}{dt}$; and, to express the former in terms of the latter, it is necessary to express dA in terms of dx. Since x is an independent variable, we may assume dx to be constant; the rate at which A is generated is then a variable rate, because PR or y is of variable length, while R moves at a constant rate along the axis of x. Now dA is the increment which A would receive in the time

dt, were the rate of A to become constant (see Diff. Calc., Art. 22). If, now, at the instant when the ordinate passes the position PR in the figure, its length should become constant, the rate of the area would become constant, and the increment which would then be received in the time *dt*, namely, the rectangle $PQSR$, represents dA. Since the base RS of this rectangle is dx, we have

$$dA = ydx = f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, by the definition given in Art. 2, A is an integral, and is denoted by

$$A = \int f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Definite Integrals.

4. Equation (2) expresses that A is a function of x, whose differential is $f(x)dx$; this function, like that considered in Art. 2, involves an undetermined constant. In fact, the expression $\int f(x)dx$ is manifestly insufficient to represent precisely the area $ABPR$, because OA, *the initial value of* x, is not indicated. The indefinite character of this expression is removed by writing this value as a subscript to the integral sign, thus, denoting the initial value by a, we write

$$A = \int_a f(x)dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the subscript is *that value of* x *for which the integral has the value zero.*

If we denote the *final value* of x (OC in the figure) by b, the area $ABDC$, which is a particular value of A, is denoted by

writing this value of x at the top of the integral sign, thus,

$$ABDC = \int_a^b f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This last expression is called a *definite integral*, and a and b are called its *limits*. In contradistinction, the expression $\int f(x)dx$ is called an *indefinite integral*.

5. As an application of the general expressions given in the last two articles, let the given curve be the parabola

$$y = x^2.$$

Equation (2) becomes in this case

$$A = \int x^2 dx.$$

Now, since $\frac{1}{3}x^3$ is a function whose differential is x^2dx, this equation gives

$$A = \int x^2 dx = \tfrac{1}{3}x^3 + C, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which C is undetermined.

Now let us suppose the limiting ordinates of the required area to be those corresponding to $x = 1$ and $x = 3$. The variable area of which we require a special value is now represented by $\int_1 x^2 dx$, which denotes that value of the indefinite integral which vanishes when $x = 1$. If we put $x = 1$ in the general expression in equation (1), namely $\frac{1}{3}x^3 + C$, we have $\frac{1}{3} + C$; hence if we subtract this quantity from the general expression, we shall have an expression which becomes zero when $x = 1$. We thus obtain

$$A = \int_1 x^2 dx = \tfrac{1}{3}x^3 - \tfrac{1}{3}.$$

Finally, putting, in this expression for the variable area, $x = 3$, we have for the required area

$$\int_1^3 x^2 dx = \tfrac{1}{3}3^3 - \tfrac{1}{3} = 8\tfrac{2}{3}.$$

6. It is evident that the definite integral obtained by this process is simply *the difference between the values of the indefinite integral at the upper and lower limits.* This difference may be expressed by attaching the limits to the symbol] affixed to the value of the indefinite integral. Thus the process given in the preceding article is written thus,

$$\int_1^3 x^2 dx = \tfrac{1}{3}x^3 + C\Big]_1^3 = 9 - \tfrac{1}{3} = 8\tfrac{2}{3}.$$

The essential part of this process is the determination of the indefinite integral or function whose differential is equal to the given expression. This is called the *integration* of the given differential expression.

Elementary Theorems.

7. *A constant factor may be transferred from one side of the integral sign to the other.* In other words, if m is a constant and u a function of x,

$$\int mu dx = m \int u dx.$$

Since each member of this equation involves an arbitrary constant, the equation only implies that the two members have the same differential. The differential of an integral is by definition the quantity under the integral sign. Now the second member is the product of a constant by a variable factor; hence its differential is $m\,d\left[\int u\,dx\right]$, that is, $m\,u\,dx$, which is also the differential of the first member.

8. This theorem is useful not only in removing constant factors from under the integral sign, but also in introducing such factors when desired. Thus, given the integral

$$\int x^n\,dx;$$

recollecting that

$$d(x^{n+1}) = (n+1)x^n\,dx,$$

we introduce the constant factor $n+1$ under the integral sign; thus,

$$\int x^n\,dx = \frac{1}{n+1}\int (n+1)x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C.$$

9. *If a differential expression be separated into parts, its integral is the sum of the integrals of the several parts.* That is, if u, v, w, $\cdots$ are functions of x,

$$\int (u+v+w+\cdots)dx = \int u\,dx + \int v\,dx + \int w\,dx + \cdots$$

For, since the differential of a sum is the sum of the differentials of the several parts, the differential of the second member is identical with that of the first member, and each member involves an arbitrary constant

Thus, for example,

$$\int (2 - \sqrt{x})\,dx = \int 2dx - \int x^{\frac{1}{2}}dx = 2x - \tfrac{2}{3}x^{\frac{3}{2}} + C,$$

the last term being integrated by means of the formula deduced in Art. 8.

Fundamental Integrals.

10. The integrals whose values are given below are called the *fundamental integrals.* The constants of integration are generally omitted for convenience.

Formula (*a*) is given in two forms, the first of which is derived in Art. 8, while the second is simply the result of putting $n = -m$. It is to be noticed that this formula gives an indeterminate result when $n = -1$; but in this case, formula (*b*) may be employed.*

The remaining formulæ are derived directly from the formulæ for differentiation; except that (*j'*), (*k'*), (*l'*), and (*m'*) are derived from (*j*), (*k*), (*l*), and (*m*) by substituting $\frac{x}{a}$ for x.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \qquad \int \frac{dx}{x^m} = -\frac{1}{(m-1)\,x^{m-1}} \quad . \quad . \quad . \quad (a)$$

$$\int \frac{dx}{x} = \log(\pm x)\dagger \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$\int a^x dx = \frac{a^x}{\log a} \qquad \int e^x dx = e^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

$$\int \cos\theta\, d\theta = \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$\int \sin\theta\, d\theta = -\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

* Applying formula (*a*) to the definite integral $\int_a^b x^n dx$, we have

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1},$$

which takes the form $\frac{0}{0}$ when $n = -1$; but, evaluating in the usual manner,

$$\left.\frac{b^{n+1} - a^{n+1}}{n+1}\right]_{n=-1} = \left.\frac{b^{n+1}\log b - a^{n+1}\log a}{1}\right]_{n=-1} = \log b - \log a;$$

a result identical with that obtained by employing formula (*b*).

† That sign is to be employed which makes the logarithm real. See Diff. Calc., Art. 60.

$$\int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \, d\theta = \tan\theta \quad \ldots\ldots\ldots\ldots \quad (f)$$

$$\int \frac{d\theta}{\sin^2\theta} = \int \operatorname{cosec}^2\theta \, d\theta = -\cot\theta \quad \ldots\ldots\ldots\ldots \quad (g)$$

$$\int \frac{\sin\theta \, d\theta}{\cos^2\theta} = \int \sec\theta \tan\theta \, d\theta = \sec\theta \quad \ldots\ldots\ldots\ldots \quad (h)$$

$$\int \frac{\cos\theta \, d\theta}{\sin^2\theta} = \int \operatorname{cosec}\theta \cot\theta \, d\theta = -\operatorname{cosec}\theta \quad \ldots\ldots\ldots\ldots \quad (i)$$

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x + C = -\cos^{-1}x + C' \quad \ldots\ldots\ldots\ldots \quad (j)$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + C = -\cos^{-1}\frac{x}{a} + C' \quad \ldots\ldots\ldots\ldots \quad (j')$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + C = -\cot^{-1}x + C' \quad \ldots\ldots\ldots\ldots \quad (k)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C' \quad \ldots\ldots\ldots\ldots \quad (k')$$

$$\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1}x + C = -\operatorname{cosec}^{-1}x + C' \quad \ldots\ldots\ldots\ldots \quad (l)$$

$$\int \frac{dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + C' \quad \ldots\ldots\ldots\ldots \quad (l')$$

$$\int \frac{dx}{\sqrt{(2x-x^2)}} = \operatorname{vers}^{-1}x \quad \ldots\ldots\ldots\ldots \quad (m)$$

$$\int \frac{dx}{\sqrt{(2ax-x^2)}} = \operatorname{vers}^{-1}\frac{x}{a} \quad \ldots\ldots\ldots\ldots \quad (m')$$

Examples I.

Find the values of the following integrals:

1. $\int \frac{dx}{\sqrt{x}}$, $2\sqrt{x}$

2. $\int \frac{dx}{x^2}$, $-\frac{1}{x}$.

3. $\int_1 \frac{dx}{x^{\frac{3}{2}}}$, $2 - \frac{2}{\sqrt{x}}$.

4. $\int_1^\infty \frac{dx}{x^5}$, $\frac{1}{4}$.

5. $\int_0 \sqrt{x}dx$, $\frac{2}{3}x^{\frac{3}{2}}$.

6. $\int_1 (x-1)^2 dx$, $\frac{x^3}{3} - x^2 + x - \frac{1}{3}$.

7. $\int_0^{\frac{a}{b}} (a-bx)^2 dx$, $a^2x - abx + \frac{b^2x^3}{3}\Big]_0^{\frac{a}{b}} = \frac{a^3}{3b}$.

8. $\int_{-a}^{a} (a+x)^3 dx$, $a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}\Big]_{-a}^{a} = 4a^4$.

9. $\int_1^{a^2} \frac{dx}{x}$, $2 \log a$.

10. $\int_{-1}^{-2} \frac{dx}{x}$, $\log(-x)\Big]_{-1}^{-2} = \log 2$.

11. $\int_a^{4a} \frac{(a+x)^2}{\sqrt{x}}\,dx$, $\qquad 2\sqrt{x}(a^2 + \frac{2}{3}ax + \frac{1}{5}x^2)\Big]_a^{4a} = 23\frac{11}{15} \cdot a^{\frac{5}{2}}$.

12. $\int_0^y e^x\,dx$, $\qquad e^y - 1$.

13. $\int_0 \sin\theta\,d\theta$, $\qquad 1 - \cos\theta$.

14. $\int_0^{\pi} \cos x\,dx$, $\qquad \sin x\Big]_0^{\pi} = 0$.

15. $\int_0^{\frac{1}{4}\pi} \frac{d\theta}{\cos^2\theta}$, $\qquad \tan\theta\Big]_0^{\frac{1}{4}\pi} = 1$.

16. $\int_0^{\frac{1}{2}a} \frac{dx}{\sqrt{(a^2 - x^2)}}$, $\qquad \sin^{-1}\frac{x}{a}\Big]_0^{\frac{1}{2}a} = \frac{\pi}{6}$.

17. $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$, $\qquad \frac{\pi}{a}$.

18 $\int_1^{\infty} \frac{dx}{x\sqrt{(x^2 - 1)}}$ $\qquad \frac{\pi}{2}$.

19 If a body is projected vertically upward, its velocity after t units of time is expressed by

$$v = a - gt,$$

a denoting the initial velocity; find the space s_1 described in the time t_1 and the greatest height to which the body will rise.

$$s_1 = \int_0^{t_1} v\,dt = at_1 - \tfrac{1}{2}gt_1^2,$$

when $v = 0$, $t = \frac{a}{g}$, $s = \frac{a^2}{2g}$.

20. If the velocity of a pendulum is expressed by

$$v = a \cos \frac{\pi t}{2\tau},$$

the position corresponding to $t = 0$ being taken as origin, find an expression for its position s at the time t, and the extreme positive and negative values of s.

$$s = \frac{2\tau a}{\pi} \sin \frac{\pi t}{2\tau},$$

$$s = \pm \frac{2\tau a}{\pi} \text{ when } t = \tau, 3\tau, 5\tau, \text{ etc.}$$

21. Find the area included between the axis of x and a branch of the curve

$$y = \sin x. \qquad 2.$$

22. Show that the area between the axis of x, the parabola

$$y^2 = 4ax,$$

and any ordinate is two thirds of the rectangle whose sides are the ordinate and the corresponding abscissa.

23. Find (α) the area included by the axes, the curve

$$y = e^x,$$

and the ordinate corresponding to $x = 1$, and (β) the whole area between the curve and axes on the left of the axis of y.

(α) $e - 1$, (β) 1.

24. Find the area between the parabola of the nth degree,

$$a^{n-1} y = x^n,$$

and the coordinates of the point (a, a). $\frac{a^2}{n+1}$.

25. Show that the area between the axis of x, the rectangular hyperbola

$$xy = 1,$$

the ordinate corresponding to $x = 1$, and any other ordinate is equivalent to the Napierian logarithm of the abscissa of the latter ordinate.

For this reason Napierian logarithms are often called hyperbolic logarithms.

26. Find the whole area between the axes, the curve

$$y^n x^m = a^{n+m},$$

and the ordinate for $x = a$, m and n being positive.

If $n > m$, $\dfrac{n a^2}{n - m}$;

if $n \leqq m$, ∞.

27. If the ordinate BR of any point B on the circle

$$x^2 + y^2 = a^2$$

be produced so that $BR \cdot RP = a^2$, prove that the whole area between the locus of P and its asymptotes is double the area of the circle.

28. Find the whole area between the axis of x and the curve

$$y(a^2 + x^2) = a^3.$$

πa^2.

29. Find the area between the axis of x and one branch of the *companion to the cycloid*, the equations of which are

$$x = a\psi \qquad y = a(1 - \cos\psi).$$

$2\pi a^2$.

II.

Direct Integration.

11. In any one of the formulæ of Art. 10, we may of course substitute for x and dx any function of x and its differential. For instance, if in formula (b) we put $x - a$ in place of x, we have

$$\int \frac{dx}{x-a} = \log (x-a) \qquad \text{or} \qquad \log (a-x),$$

according as x is greater or less than a.

When a given integral is obviously the result of such a substitution in one of the fundamental integrals, or can be made to take this form by the introduction of a constant factor, it is said to be *directly integrable*. Thus, $\int \sin mx\, dx$ is directly integrable by formula (e); for, if in this formula we put mx for θ, we have

$$\int \sin mx \cdot m\, dx = -\cos mx,$$

hence

$$\int \sin mx\, dx = \frac{1}{m} \int \sin mx \cdot m\, dx = -\frac{1}{m} \cos mx.$$

So also in $$\int \sqrt{(a + bx^2)}\, x\, dx,$$

the quantity $x\, dx$ becomes the differential of the binomial $(a + bx^2)$ when we introduce the constant factor $2b$, hence this integral can be converted into the result obtained by putting $(a + bx^2)$ in place of x in $\int \sqrt{x}\, dx$, which is a case of formula (a). Thus

$$\int \sqrt{(a + bx^2)}\, x\, dx = \frac{1}{2b} \int (a + bx^2)^{\frac{1}{2}} 2bx\, dx = \frac{1}{3b} (a + bx^2)^{\frac{3}{2}}.$$

12. A simple algebraic or trigonometric transformation sometimes suffices to render an expression directly integrable, or to separate it into directly integrable parts. Thus, since $-\sin x\,dx$ is the differential of $\cos x$, we have by formula (b)

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\log \cos x\,.$$

So also, by formula (f),

$$\int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta\,;$$

by (e) and (a),

$$\int \sin^3\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta = -\cos\theta + \tfrac{1}{3}\cos^3\theta\,;$$

by (j) and (a),

$$\int \sqrt{\left(\frac{1+x}{1-x}\right)}\,dx = \int \frac{1+x}{\sqrt{(1-x^2)}}\,dx$$

$$= \int \frac{dx}{\sqrt{(1-x^2)}} - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x\,dx) = \sin^{-1}x - \sqrt{(1-x^2)}.$$

Rational Fractions.

13. When the coefficient of dx in an integral is a fraction whose terms are rational functions of x, the integral may generally be separated into parts directly integrable. If the denominator is of the first degree, we proceed as in the following example.

Given the integral $\displaystyle\int \frac{x^2 - x + 3}{2x+1}\,dx$;

by division,

$$\frac{x^2 - x + 3}{2x+1} = \frac{x}{2} - \frac{3}{4} + \frac{15}{4}\frac{1}{2x+1},$$

hence

$$\int \frac{x^2 - x + 3}{2x + 1} dx = \frac{1}{2}\int x\, dx - \frac{3}{4}\int dx + \frac{15}{4}\int \frac{dx}{2x + 1}$$

$$= \frac{x^2}{4} - \frac{3x}{4} + \frac{15}{8}\log (2x + 1).$$

When the denominator is of higher degree, it is evident that we may, by division, make the integration depend upon that of a fraction in which the degree of the numerator is lower than that of the denominator by at least a unit. We shall consider therefore fractions of this form only.

Denominators of the Second Degree.

14. If the denominator is of the second degree, it will (after removing a constant, if necessary) either be the square of an expression of the first degree, or else such a square increased or diminished by a constant. As an example of the first case, let us take

$$\int \frac{x + 1}{(x - 1)^2} dx.$$

The fraction may be decomposed thus:

$$\frac{x + 1}{(x - 1)^2} = \frac{x - 1 + 2}{(x - 1)^2} = \frac{1}{x - 1} + \frac{2}{(x - 1)^2};$$

hence

$$\int \frac{x + 1}{(x - 1)^2} dx = \int \frac{dx}{x - 1} + 2\int \frac{dx}{(x - 1)^2}$$

$$= \log (x - 1) - \frac{2}{x - 1}.$$

15. The integral $\int \frac{x + 3}{x^2 + 2x + 6} dx$

affords an example of the second case, for the denominator may be written in the form

$$x^2 + 2x + 6 = (x + 1)^2 + 5.$$

Decomposing the fraction as in the preceding article,

$$\frac{x+3}{(x+1)^2+5} = \frac{x+1}{(x+1)^2+5} + \frac{2}{(x+1)^2+5};$$

whence

$$\int \frac{x+3}{x^2+2x+6}\,dx = \int \frac{(x+1)\,dx}{(x+1)^2+5} + 2\int \frac{dx}{(x+1)^2+5}.$$

The first of the integrals in the second member is directly integrable by formula (b), since the differential of the denominator is $2\,(x+1)\,dx$, and the second is a case of formula (k'). Therefore

$$\int \frac{x+3}{x^2+2x+6}\,dx = \tfrac{1}{2}\log(x^2+2x+6) + \frac{2}{\sqrt{5}}\tan^{-1}\frac{x+1}{\sqrt{5}}.$$

16. To illustrate the third case, let us take

$$\int \frac{2x+1}{x^2-x-6}\,dx,$$

in which the denominator is equivalent to $(x - \frac{1}{2})^2 - 6\frac{1}{4}$, and can therefore be resolved into real factors of the first degree. We can then decompose the fraction into fractions having these factors for denominators. Thus, in the present example, assume

$$\frac{2x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}, \quad \ldots\ldots \quad (1)$$

in which A and B are numerical quantities to be determined. Multiplying by $(x-3)\,(x+2)$,

$$2x + 1 = A\,(x+2) + B\,(x-3). \quad \ldots\ldots \quad (2)$$

$1 = \frac{A}{k+x} + \frac{B}{k-x}$

$1 = A(k-x) + B(k+x)$.

Since equation (2) is an algebraic identity, we may in it assign any value we choose to x. Putting $x = 3$, we find

$$7 = 5A, \qquad \text{whence} \qquad A = \tfrac{7}{5},$$

putting $x = -2$,

$$-3 = -5B, \qquad \text{whence} \qquad B = \tfrac{3}{5}.$$

Substituting these values in (1),

$$\frac{2x+1}{x^2-x-6} = \frac{7}{5(x-3)} + \frac{3}{5(x+2)},$$

whence

$$\int \frac{2x+1}{x^2-x-6}\,dx = \tfrac{7}{5}\log(x-3) + \tfrac{3}{5}\log(x+2).$$

17. If the denominator, in a case of the kind last considered, is denoted by $(x-a)(x-b)$, a and b are evidently the roots of the equation formed by putting this denominator equal to zero. The cases considered in Art. 14 and Art. 15 are respectively those in which the roots of this equation are equal, and those in which the roots are imaginary. When the roots are real and unequal, if the numerator does not contain x, the integral can be reduced to the form

$$\int \frac{dx}{(x-a)(x-b)},$$

and by the method given in the preceding article we find

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\Big[\log(x-a) - \log(x-b)\Big]$$

$$= \frac{1}{a-b}\log\frac{x-a}{x-b}, \quad \ldots\ldots \quad (A)^*$$

* The formulæ of this series are collected together at the end of Chapter II. for convenience of reference. See Art. 101.

in which, when $x < a$, $\log (a - x)$ should be written in place of $\log (x - a)$. [See note on formula (b), Art. 10.]

If $b = -a$, this formula becomes

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} \quad \ldots\ldots \quad (A')$$

Integrals of the special forms given in (A) and (A') may be evaluated by the direct application of these formulas. Thus, given the integral

$$\int \frac{dx}{2x^2 + 3x - 2};$$

if we place the denominator equal to zero, we have the roots $a = \frac{1}{2}$, $b = -2$; whence by formula (A),

$$\int \frac{dx}{2x^2 + 3x - 2} = \tfrac{1}{2} \int \frac{dx}{(x - \frac{1}{2})(x + 2)} = \frac{1}{2} \cdot \frac{1}{2\frac{1}{2}} \log \frac{x - \frac{1}{2}}{x + 2};$$

or, since $\log (2x - 1)$ differs from $\log (x - \frac{1}{2})$ only by a constant, we may write

$$\int \frac{dx}{2x^2 + 3x - 2} = \frac{1}{5} \log \frac{2x - 1}{x + 2}.$$

Denominators of Higher Degree.

18. When the denominator is of a degree higher than the second, we may in like manner suppose it resolved into factors corresponding to the roots of the equation formed by placing it equal to zero. The fraction (of which we suppose the numerator to be lower in degree than the denominator) may now be decomposed into partial fractions. If the roots are all real and unequal, we assume these partial fractions as in Art. 16; there being one assumed fraction for each factor.

If, however, a pair of imaginary roots occurs, the factor cor-

responding to the pair is of the form $(x-\alpha)^2+\beta^2$, and the partial fraction must be assumed in the form

$$\frac{Ax+B}{(x-\alpha)^2+\beta^2};$$

for we are only entitled to assume that the numerator of each partial fraction is lower in degree than its denominator (otherwise the given fraction which is the sum of the partial fractions would not have this property). For example, given

$$\int\frac{x+3}{(x^2+1)(x-1)}dx.$$

Assume

$$\frac{x+3}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}, \quad . \quad . \quad . \quad (1)$$

whence

$$x+3=(x-1)(Ax+B)+(x^2+1)C.$$

Since in an identical equation the coefficients of the several powers of x must separately vanish, the coefficients of x^2, x and x^0 give, for the determination of A, B and C, the three equations

$$A+C=0, \qquad B-A=1, \qquad C-B=3.$$

From these we obtain $A=-2$, $B=-1$, $C=2$; hence, substituting in equation (1),

$$\frac{x+3}{(x^2+1)(x-1)}=\frac{2}{x-1}-\frac{2x+1}{x^2+1},$$

therefore

$$\int\frac{x+3}{(x^2+1)(x-1)}dx=2\int\frac{dx}{x-1}-\int\frac{2x\,dx}{x^2+1}-\int\frac{dx}{x^2+1}$$
$$=2\log(x-1)-\log(x^2+1)-\tan^{-1}x.$$

19. The method of determining the assumed coefficients illustrated above makes it evident that, the denominator being

of the nth degree, we must assume n of these coefficients; because we have to satisfy n equations, derived from the powers of x from x^{n-1} to x^0.

It is evident that we may take for the denominators of the partial fractions any two or more factors of the given denominator which have no common factor between themselves, provided we assume for each numerator a polynomial of degree just inferior to that of the denominator. But, since our present object is to separate the given fraction into directly integrable parts, when a squared factor such as $(x-a)^2$ occurs, instead of assuming the corresponding partial fraction in the form $\frac{Ax+B}{(x-a)^2}$ (which would require to be further decomposed as in Art. 14), we at once assume a pair of fractions of the form

$$\frac{A}{x-a}+\frac{B}{(x-a)^2}.$$

20. We proceed in like manner when a higher power of a linear factor occurs. For example, given

$$\int\frac{x+2}{(x-1)^3(x+1)}dx;$$

we assume

$$\frac{x+2}{(x-1)^3(x+1)}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{x+1};$$

whence

$$x+2=[A+B(x-1)+C(x-1)^2](x+1)+D(x-1)^3. \quad (1)$$

Putting $x=1$, we have $\quad 3=2A \quad \therefore \quad A=\frac{3}{2}$.

Putting $x=-1$, we have $\quad 1=-8D \quad \therefore \quad D=-\frac{1}{8}$.

The most convenient way to determine the other coefficients is to equate to zero the coefficient of x^3, and to put $x=0$. We thus obtain

$$0 = C + D, \quad \text{and} \quad 2 = A - B + C - D,$$

from which $C = \frac{1}{8}$, and $B = -\frac{1}{4}$. Therefore

$$\int \frac{x+2}{(x-1)^3(x+1)}dx = \frac{3}{2}\int \frac{dx}{(x-1)^3} - \frac{1}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{8}\int \frac{dx}{x-1} - \frac{1}{8}\int \frac{dx}{x+1}$$

$$= -\frac{3}{4(x-1)^2} + \frac{1}{4(x-1)} + \frac{1}{8}\log\frac{x-1}{x+1}.$$

21. The fraction corresponding to a simple factor of the given denominator may be found, independently of the other partial fractions, by means of the expression derived below. Denote the given denominator by $\phi(x)$, and let it contain the simple factor $x - a$, so that

$$\phi(x) = (x-a)\psi(x), \quad \ldots\ldots \quad (1)$$

in which $\psi(x)$ does not vanish when $x = a$.

Let $f(x)$ denote the numerator (it is not necessary here to suppose this to be lower than $\phi(x)$ in degree), and let Q denote the entire part of the quotient. Then we may assume

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x-a)\psi(x)} = Q + \frac{A}{x-a} + \frac{P}{\psi(x)},$$

in which Q and P are in general polynomials. Clearing of fractions,

$$f(x) = Q(x-a)\psi(x) + A\psi(x) + P(x-a).$$

Putting $x = a$, we have (since neither P nor Q can become infinite)

$$f(a) = A\psi(a), \quad \text{whence} \quad A = \frac{f(a)}{\psi(a)}, \quad \ldots \quad (2)$$

an expression for the numerator A.

Again, differentiating equation (1), we have

$$\phi'(x) = \psi(x) + (x-a)\psi'(x);$$

whence, putting $x = a$, $\phi'(a) = \psi(a)$, the substitution of which in equation (2) gives another expression for A, namely,

$$A = \frac{f(a)}{\phi'(a)}. \quad \ldots \ldots \ldots \quad (3)$$

As an example, let us find the value of

$$\int \frac{x^5 + 2}{x^4 + 2x^3 - x^2 - 2x}\, dx.$$

The denominator is the product $x(x^2 - 1)(x + 2)$, and the first term of the quotient is obviously x; hence we assume

$$\frac{x^5 + 2}{x^4 + 2x^3 - x^2 - 2x} = x + \alpha + \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x+2}.$$

The coefficient of x^4 in the equation cleared of fractions gives $\alpha = -2$. Now forming the fraction

$$\frac{f(x)}{\phi'(x)} = \frac{x^5 + 2}{4x^3 + 6x^2 - 2x - 2},$$

we may determine A, B, C and D in accordance with expression (3) by putting therein for x successively 0, -1, 1 and -2. Thus $A = -1$, $B = \frac{1}{2}$, $C = \frac{1}{2}$, $D = 5$; whence

$$\frac{x^5 + 2}{x^4 + 2x^3 - x^2 - 2x} = x - 2 - \frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)} + \frac{5}{x+2},$$

and

$$\int \frac{x^5 + 2}{x^4 + 2x^3 - x^2 - 2x}\, dx = \frac{x^2}{2} - 2x + \log \frac{(x+2)^5 \sqrt{(x^2 - 1)}}{x}.$$

22. We have seen that the decomposition of a given fraction into partial fractions presupposes a knowledge of the roots of the equation resulting from equating the given denominator to zero. In the case of the denominator $x^n - 1$, we can employ the expressions for the imaginary roots involving the circular functions of certain angles. (Diff. Calc., Art. 231). In some simple cases the factors are expressible in ordinary surds. For example, we have

$$x^8 - 1 = (x - 1)(x + 1)(x^2 + 1)(x^2 - x\sqrt{2} + 1)(x^2 + x\sqrt{2} + 1).$$

It is to be noticed that when the fraction is a rational function of some higher power of x, we may simplify the process of decomposition. Thus it is legitimate to assume

$$\frac{x^4}{x^8 - 1} = \frac{A}{x^4 - 1} + \frac{B}{x^4 + 1},$$

because, if $z = x^4$, the numerator and factors of the denominator are rational functions of z. The first term could be treated in like manner with respect to x^2; but not the second, since the factors of $x^4 + 1$ involve x as well as x^2,

23. We have seen in the preceding articles that the partial fractions corresponding to the real roots, whether single or multiple, are directly integrable, and also those corresponding to unrepeated imaginary roots. In the following example a case of multiple imaginary roots occurs: given

$$\int \frac{a + x}{(a - x)(a^2 + x^2)^2}\, dx.$$

It is readily shown that the partial fraction corresponding to $a - x$ is $\frac{1}{2a^3(a - x)}$, and the remainder is conveniently found by subtracting this from the given fraction; thus

$$\frac{a + x}{(a - x)(a^2 + x^2)^2} - \frac{1}{2a^3(a - x)} = \frac{a^4 + 2a^3x - 2a^2x^2 - x^4}{2a^3(a - x)(a^2 + x^2)^2}.$$

The numerator of this remainder is found to be divisible by $a - x$, thus verifying the process; and the given integral reduces to

$$-\frac{1}{2a^3}\log(a - x) + \frac{1}{2a^3}\int \frac{a^3 + 3a^2x + ax^2 + x^3}{(a^2 + x^2)^2}\, dx.$$

The integration of the last term may be effected by a transformation given later. See Art. 41 and Art. 76.

24. Instead of assuming the partial fractions with undeter-

mined numerators, it is sometimes possible to proceed more expeditiously as in the following examples:

Given

$$\int \frac{1}{x^3(1+x^2)}\,dx;$$

putting the numerator in the form $1 + x^2 - x^2$, we have

$$\int \frac{1}{x^3(1+x^2)}\,dx = \int \frac{1+x^2}{x^3(1+x^2)}\,dx - \int \frac{x^2}{x^3(1+x^2)}\,dx$$

$$= \int \frac{dx}{x^3} - \int \frac{1}{x(1+x^2)}\,dx.$$

Treating the last integral in like manner,

$$\int \frac{1}{x^3(1+x^2)}\,dx = \int \frac{dx}{x^3} - \int \frac{dx}{x} + \int \frac{x\,dx}{1+x^2}$$

$$= -\frac{1}{2x^2} - \log x + \tfrac{1}{2}\log(1+x^2) = -\frac{1}{2x^2} + \log \frac{\sqrt{(1+x^2)}}{x}.$$

Again, given

$$\int \frac{1}{x^2(1+x)^2}\,dx;$$

putting the numerator in the form $(1+x)^2 - 2x - x^2$, we have

$$\int \frac{1}{x^2(1+x)^2}\,dx = \int \frac{dx}{x^2} - \int \frac{2+x}{x(1+x)^2}\,dx$$

$$= \int \frac{dx}{x^2} - 2\int \frac{dx}{x(1+x)^2} + \int \frac{dx}{(1+x)^2}.$$

Hence by equation (*A*), Art. 17,

$$\int \frac{dx}{x^2(1+x)^2} = -\frac{1}{x} - 2\log\frac{x}{1+x} - \frac{1}{1+x}.$$

Examples II.

1. $\int \frac{dx}{a-x}$, $\quad -\log(a-x)$.

2. $\int \frac{dx}{(a-x)^2}$, $\quad \frac{1}{a-x}$.

3. $\int \frac{x\,dx}{a^2+x^2}$, $\quad \frac{1}{2}\log(a^2+x^2)$.

4. $\int_0 \sqrt{(a^2-x^2)}\,x\,dx$, $\quad \frac{a^3-(a^2-x^2)^{\frac{3}{2}}}{3}$.

5. $\int_0 \frac{x^2\,dx}{a^3-x^3}$, $\quad \frac{1}{3}\log\frac{a^3}{a^3-x^3}$.

6. $\int_0 \frac{x\,dx}{\sqrt{(a^2-x^2)}}$, $\quad a-\sqrt{(a^2-x^2)}$.

7. $\int (a^2+3x^2)^3\,x\,dx$, $\quad \frac{(a^2+3x^2)^4}{24}$.

8. $\int_0 (a+mx)^2\,dx$, $\quad \frac{(a+mx)^3-a^3}{3m}$.

9. $\int \frac{dx}{\sin^2 2x}$, $\quad -\frac{\cot 2x}{2}$.

10. $\int_0 \cos^3 x \sin x\,dx$, $\quad \frac{1-\cos^4 x}{4}$.

11. $\int \frac{\cos\theta\,d\theta}{\sin^3\theta}$, $\quad -\frac{1}{2}\operatorname{cosec}^2\theta$.

12. $\int_0 \sec^3 3x \tan 3x\,dx$, $\quad \frac{\sec^3 3x-1}{9}$.

13. $\int a^{mx} dx,$ $\qquad \frac{a^{mx}}{m \log a}.$

14. $\int (e^x - 1)^3 dx,$ $\qquad \frac{1}{3}e^{3x} - \frac{3}{2}e^{2x} + 3e^x - x.$

15. $\int (1 + 3\sin^2 x)^2 \sin x \cos x \, dx,$ $\qquad \frac{(1 + 3\sin^2 x)^3}{18}.$

16. $\int_0^{2a} \frac{(a - x)\, dx}{\sqrt{(2ax - x^2)}},$ $\qquad \sqrt{(2ax - x^2)}\Big]_0^{2a} = 0.$

17. $\int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta,$ $\qquad \frac{2}{3}.$

18. $\int \sec^4\theta \, d\theta,$ $\qquad \tan\theta + \frac{1}{3}\tan^3\theta.$

19. $\int \tan^3 x \, dx,$ $\qquad \frac{1}{2}\tan^2 x + \log\cos x.$

20. $\int_0^{\frac{\pi}{4}} \sec^4 x \tan x \, dx,$ $\qquad \frac{1}{4}\sec^4 x\Big]_0^{\frac{\pi}{4}} = \frac{3}{4}.$

21. $\int \sqrt{\frac{a - x}{a + x}} dx,$ $\qquad a\sin^{-1}\frac{x}{a} + \sqrt{(a^2 - x^2)}.$

22. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3\theta \, d\theta,$ $\qquad \frac{1 - \log 2}{2}.$

23. $\int \sqrt{\frac{2a - x}{x}} dx,$ $\qquad \sqrt{(2ax - x^2)} + a\,\mathrm{vers}^{-1}\frac{x}{a}.$

24. $\int \sin(\alpha - 2\theta)\, d\theta,$ $\qquad \frac{\cos(\alpha - 2\theta)}{2}.$

25. $\int \frac{\cos x\, dx}{a - b \sin x}$, $\quad -\frac{1}{b}\log(a - b\sin x)$.

26. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\tan x}$, $\quad \frac{1}{2}\log 2$.

27. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\tan x}$, $\quad \frac{1}{2}\log 2$.

28. $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{x \log x}$, $\quad \log(-\log x)\Big]_{\frac{1}{4}}^{\frac{1}{2}} = -\log 2$.

29. $\int \frac{dx}{e^x + e^{-x}}$, $\quad \tan^{-1} e^x$.

30. $\int \frac{x^2\, dx}{x^6 + 1}$, $\quad \frac{1}{3}\tan^{-1} x^3$.

31. $\int \frac{x\, dx}{\sqrt{(a^4 - x^4)}}$, $\quad \frac{1}{2}\sin^{-1}\frac{x^2}{a^2}$.

32. $\int \frac{dx}{\sqrt{(5 - 3x^2)}}$, $\quad \frac{1}{\sqrt{3}}\sin^{-1}\frac{x\sqrt{3}}{\sqrt{5}}$.

33. $\int \frac{dx}{2 + 5x^2}$, $\quad \frac{1}{\sqrt{10}}\tan^{-1}\frac{x\sqrt{5}}{\sqrt{2}}$.

34. $\int_1^\infty \frac{dx}{x\sqrt{(2x^2 - 1)}}$, $\quad \frac{\pi}{4}$.

35. $\int_0^1 \frac{dx}{x^2 + x + 1}$, $\quad \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}}\Big]_0^1 = \frac{\pi}{3\sqrt{3}}$.

36. $\int_0^1 \frac{dx}{\sqrt{(5 - 4x - x^2)}}$, $\cos^{-1}\frac{2}{3}$.

37. $\int \frac{\sqrt{(x^2 - a^2)}}{x}\,dx \left[= \int \frac{x^2 - a^2}{x\sqrt{(x^2 - a^2)}}\,dx\right]$,

$\sqrt{(x^2 - a^2)} - a \sec^{-1}\frac{x}{a}$.

38. $\int_0^{\frac{1}{2}a} \frac{x^3}{a - x}\,dx$, $a^2(\log 2 - \frac{5}{8})$.

39. $\int \frac{4x^3 - x + 3}{x^2 + 1}\,dx$, $4x - \frac{1}{2}\log(x^2 + 1) - \tan^{-1}x$.

40. $\int \frac{x^2 + x + 1}{x^2 - x + 1}\,dx$, $x + \log(x^2 - x + 1) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x - 1}{\sqrt{3}}$.

41. $\int \frac{x^2 - 1}{x^2 - 4}\,dx$, $x + \frac{3}{4}\log\frac{x - 2}{x + 2}$.

42. $\int \frac{(1 + x)^2}{x - x^2}\,dx$, $\log\frac{x}{(1 - x)^4} - x$.

43. $\int \frac{(2x + 1)^2\,dx}{2x + 3}$, $x^2 - x + 2\log(2x + 3)$.

44. $\int \frac{2x + 3}{(2x + 1)^2}\,dx$, $\frac{1}{2}\log(2x + 1) - \frac{1}{2x + 1}$.

45. $\int \frac{x^2 - 3x + 3}{x^2 - 3x + 2}\,dx$, $x + \log\frac{x - 2}{x - 1}$.

46. $\int_0^a \frac{dx}{x^2 - 2ax\cos\alpha + a^2}$,

$$\frac{1}{a\sin\alpha}\tan^{-1}\frac{x - a\cos\alpha}{a\sin\alpha}\Big]_0^a = \frac{\pi - \alpha}{2a\sin\alpha}.$$

47. $\int \frac{dx}{x^2 - 2ax\sec\alpha + a^2}$, $\quad \frac{1}{2a\tan\alpha}\log\frac{x - a\sec\alpha - a\tan\alpha}{x - a\sec\alpha + a\tan\alpha}$.

48. $\int \frac{dx}{2x^2 - 4x - 7}$, $\quad \frac{\sqrt{2}}{12}\log\frac{2x - 2 - 3\sqrt{2}}{2x - 2 + 3\sqrt{2}}$.

49. $\int \frac{x^2\,dx}{1 - x^6}$, $\quad \frac{1}{6}\log\frac{1 + x^3}{1 - x^3}$.

50. $\int \frac{3x - 1}{x^3 - x^2 - 2x}\,dx$, $\quad \frac{1}{6}\log\frac{x^3(x - 2)^5}{(x + 1)^5}$.

51. $\int \frac{x\,dx}{(x + 2)(x + 3)^2}$, $\quad 2\log\frac{x + 3}{x + 2} - \frac{3}{x + 3}$.

52. $\int \frac{x\,dx}{x^3 + x^2 + x + 1}$, $\quad \frac{1}{2}\left[\tan^{-1}x + \log\frac{\sqrt{(x^2 + 1)}}{x + 1}\right]$.

53. $\int \frac{x^2 dx}{x^4 + x^2 - 2}$, $\quad \frac{1}{6}\log\frac{x - 1}{x + 1} + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}$.

54. $\int \frac{x^2 - x + 2}{x^4 - 5x^2 + 4}\,dx$, $\quad \frac{2}{3}\log\frac{x + 1}{x + 2} + \frac{1}{3}\log\frac{x - 2}{x - 1}$.

55. $\int \frac{dx}{x^3 - x^2 - x + 1}$, $\quad \frac{1}{4}\log\frac{x + 1}{x - 1} - \frac{1}{2(x - 1)}$.

56. $\int \frac{3x + 1}{x^4 - 1}\,dx$, $\quad \log\frac{(x - 1)\sqrt{(x + 1)}}{(x^2 + 1)^{\frac{3}{4}}} - \frac{1}{2}\tan^{-1}x$.

57. $\int \frac{dx}{1 + x^3}$, $\quad \frac{1}{6}\log\frac{(x + 1)^2}{x^2 - x + 1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x - 1}{\sqrt{3}}$.

58. $\int \frac{x^2 dx}{(x - 1)^2(x^2 + 1)}$, $\quad \frac{1}{2}\log(x - 1) - \frac{1}{4}\log(x^2 + 1) - \frac{1}{2(x - 1)}$.

59. $\int \frac{dx}{x(1+x+x^2+x^3)}$,

$$\log x - \frac{1}{2}\log(1+x) - \frac{1}{4}\log(1+x^2) - \frac{1}{2}\tan^{-1}x.$$

60. $\int \frac{x^2-1}{x^4+x^2+1}\,dx$, $\qquad \frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}$.

61. $\int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$, $\qquad \frac{1}{6}\log x + \frac{1}{2}\log(x-2) + \frac{1}{3}\log(x+3)$.

62. $\int \frac{x^2dx}{x^4-x^2-12}$, $\qquad \frac{1}{7}\log\frac{x-2}{x+2} + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}$.

63. $\int \frac{x^2dx}{(x^2-1)^2}$, $\qquad \frac{1}{4}\log\frac{x-1}{x+1} - \frac{x}{2(x^2-1)}$.

64. $\int \frac{2x^2-3a^2}{x^4-a^4}\,dx$, $\qquad \frac{5}{2a}\tan^{-1}\frac{x}{a} - \frac{1}{4a}\log\frac{x-a}{x+a}$.

65. $\int \frac{x\,dx}{x^4-x^2-2}$, $\qquad \frac{1}{6}\log\frac{x^2-2}{x^2+1}$.

66. $\int \frac{dx}{(x^2+a^2)(x+b)}$, $\qquad \frac{1}{b^2+a^2}\left[\log\frac{x+b}{\sqrt{(x^2+a^2)}} + \frac{b}{a}\tan^{-1}\frac{x}{a}\right]$.

67. $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $\qquad \frac{\pi}{2ab(a+b)}$.

68. $\int_a^\infty \frac{a^2dx}{x^2(a^2+x^2)}$, $\qquad \frac{4-\pi}{4a}$.

69. $\int \frac{x+1}{x(1+x^2)}\,dx$, $\qquad \tan^{-1}x + \log\frac{x}{\sqrt{(1+x^2)}}$.

70. $\int \frac{dx}{x^4(x^2+1)}$, $\qquad \tan^{-1}x + \frac{1}{x} - \frac{1}{3x^3}$.

71. $\int \frac{dx}{x(1+x)^2}$, $\log \frac{x}{1+x} + \frac{1}{1+x}$.

72. $\int \frac{dx}{x(a+bx^3)}$, $\frac{1}{3a} \log \frac{x^3}{a+bx^3}$.

73. $\int \frac{dx}{x^4(a+bx^3)}$, $-\frac{1}{3ax^3} + \frac{b}{3a^2} \log \frac{a+bx^3}{x^3}$.

74. Find the whole area enclosed by both loops of the curve

$$y^2 = x^2(1-x^2).$$ $\frac{4}{3}$.

75. Find the area enclosed between the asymptote corresponding to $x = a$, and the curve

$$x^2y^2 + a^2x^2 = a^2y^2.$$ $2a$.

76. Find the whole area enclosed by the curve

$$a^2y^4 = x^4(a^2 - x^2).$$ $\frac{8}{5}a^2$.

77. Find the area enclosed by the catenary

$$y = \frac{c}{2}\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right],$$

the axes and any ordinate.

$$\frac{c^2}{2}\left[e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right].$$

78. Find the whole area between the witch

$$xy^2 = 4a^2(2a - x)$$

and its asymptote. *See Ex.* 23.

$4\pi a^2$.

III.

Trigonometric Integrals.

25. The transformation, $\tan^2\theta = \sec^2\theta - 1$, suffices to separate all integrals of the form

$$\int \tan^n \theta \, d\theta, \quad \ldots\ldots\ldots \quad (1)$$

in which n is an integer, into directly integrable parts. Thus, for example,

$$\int \tan^5 \theta \, d\theta = \int \tan^3 \theta \, (\sec^2 \theta - 1) \, d\theta$$

$$= \frac{\tan^4 \theta}{4} - \int \tan^3 \theta \, d\theta.$$

Transforming the last integral in like manner, we have

$$\int \tan^5 \theta \, d\theta = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + \int \tan \theta \, d\theta;$$

hence (see Art. 12)

$$\int \tan^5 \theta \, d\theta = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} - \log \cos \theta.$$

When the value of n in (1) is even, the value of the final integral will be θ. When n is negative, the integral takes the form

$$\int \cot^n \theta \, d\theta,$$

which may be treated in a similar manner.

26. Integrals of the form

$$\int \sec^n \theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are readily evaluated *when* n *is an even number*, thus

$$\int \sec^6 \theta \, d\theta = \int (\tan^2 \theta + 1)^2 \sec^2 \theta \, d\theta$$

$$= \int \tan^4 \theta \sec^2 \theta \, d\theta + 2 \int \tan^2 \theta \sec^2 \theta \, d\theta + \int \sec^2 \theta \, d\theta$$

$$= \frac{\tan^5 \theta}{5} + \frac{2 \tan^3 \theta}{3} + \tan \theta.$$

If n in expression (2) is odd, the method to be explained in Section VI is required.

Integrals of the form $\int \operatorname{cosec}^n \theta \, d\theta$ are treated in like manner.

Cases in which $\sin^m \theta \cos^n \theta \, d\theta$ *is directly integrable.*

27. If n is a *positive odd number*, an integral of the form

$$\int \sin^m \theta \cos^n \theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is directly integrable in terms of $\sin \theta$. Thus,

$$\int \sin^2 \theta \cos^5 \theta \, d\theta = \int \sin^2 \theta \, (1 - \sin^2 \theta)^2 \cos \theta \, d\theta$$

$$= \frac{\sin^3 \theta}{3} - \frac{2 \sin^5 \theta}{5} + \frac{\sin^7 \theta}{7}.$$

This method is evidently applicable even when m is fractional or negative. Thus, putting y for $\sin \theta$,

$$\int \frac{\cos^3\theta}{\sin^{\frac{3}{2}}\theta}\,d\theta = \int \frac{(1-y^2)\,dy}{y^{\frac{3}{2}}} = \int y^{-\frac{3}{2}}\,dy - \int y^{\frac{1}{2}}\,dy;$$

hence

$$\int \frac{\cos^3\theta}{\sin^{\frac{3}{2}}\theta}\,d\theta = -2y^{-\frac{1}{2}} - \frac{2}{3}y^{\frac{3}{2}} = -\frac{2}{3}\cdot\frac{3+\sin^2\theta}{\sqrt{(\sin\theta)}}.$$

When m in expression (3) is a positive odd number, the integral is evaluated in a similar manner.

28. An integral of the form (3) is also directly integrable *when* m + n *is an even negative integer*, in other words, when it can be written in the form

$$\int \frac{\sin^m\theta\,d\theta}{\cos^{m+2q}\theta} = \int \tan^m\theta\,\sec^{2q}\theta\,d\theta,$$

in which q *is positive.*

For example,

$$\int \frac{d\theta}{\sin^{\frac{3}{2}}\theta\,\cos^{\frac{5}{2}}\theta} = \int (\tan\theta)^{-\frac{3}{2}}\sec^4\theta\,d\theta$$

$$= \int (\tan\theta)^{-\frac{3}{2}}(\tan^2\theta + 1)\sec^2\theta\,d\theta;$$

hence

$$\int \frac{d\theta}{\sin^{\frac{3}{2}}\theta\,\cos^{\frac{5}{2}}\theta} = \frac{2}{3}\tan^{\frac{3}{2}}\theta - \frac{2}{\tan^{\frac{1}{2}}\theta}.$$

It may be more convenient to express the integral in terms of cot θ and cosec θ, thus

$$\int \frac{\cos^4\theta\,d\theta}{\sin^8\theta} = \int \cot^4\theta\,(\cot^2\theta + 1)\operatorname{cosec}^2\theta\,d\theta$$

$$= -\frac{\cot^7\theta}{7} - \frac{\cot^5\theta}{5}.$$

Integrals of the forms treated in Art. 25 and Art. 26 are included in the general form (3), Art. 27. Except in the cases already considered, and in the special cases given below, the method of reduction given in Section VI is required in the evaluation of integrals of this form.

The Integrals $\int \sin^2\theta\, d\theta$, *and* $\int \cos^2\theta\, d\theta$.

29. These integrals are readily evaluated by means of the transformations

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta), \qquad \text{and} \qquad \cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta).$$

Thus

$$\int \sin^2\theta\, d\theta = \tfrac{1}{2}\int d\theta - \tfrac{1}{2}\int \cos 2\theta\, d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta,$$

or, since

$$\sin 2\theta = 2 \sin\theta \cos\theta,$$

$$\int \sin^2\theta\, d\theta = \tfrac{1}{2}(\theta - \sin\theta\cos\theta). \quad . \quad . \quad . \quad . \quad (B)$$

In like manner

$$\int \cos^2\theta\, d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta). \quad . \quad . \quad . \quad . \quad (C)$$

Since $\sin^2\theta + \cos^2\theta = 1$, the sum of these integrals is $\int d\theta$; accordingly we find the sum of their values to be θ.

In the applications of the Integral Calculus, these integrals frequently occur with the limits 0 and $\frac{1}{2}\pi$; from (B) and (C) we derive

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \tfrac{1}{4}\pi.$$

The Integrals $\int \frac{d\theta}{\sin\theta\cos\theta}$, $\int \frac{d\theta}{\sin\theta}$, *and* $\int \frac{d\theta}{\cos\theta}$.

30. We have

$$\int \frac{d\theta}{\sin\theta\cos\theta} = \int \frac{\sec^2\theta\, d\theta}{\tan\theta} = \log\tan\theta. \quad . \quad . \quad . \quad (D)$$

Again, using the transformation,

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta,$$

we have

$$\int \frac{d\theta}{\sin\theta} = \int \frac{\frac{1}{2}d\theta}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \int \frac{\sec^2\frac{1}{2}\theta\,\frac{1}{2}d\theta}{\tan\frac{1}{2}\theta};$$

hence

$$\int \frac{d\theta}{\sin\theta} = \log\tan\tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

This integral may also be evaluated thus,

$$\int \frac{d\theta}{\sin\theta} = \int \frac{\sin\theta\, d\theta}{\sin^2\theta} = \int \frac{\sin\theta\, d\theta}{1-\cos^2\theta}.$$

Since $\sin\theta\, d\theta = -d(\cos\theta)$, the value of the last integral is, by formula (A'), Art. 17,

$$\frac{1}{2}\log\frac{1-\cos\theta}{1+\cos\theta} = \log\sqrt{\frac{1-\cos\theta}{1+\cos\theta}};$$

and, multiplying both terms of the fraction by $1-\cos\theta$, we have

$$\int \frac{d\theta}{\sin\theta} = \log\frac{1-\cos\theta}{\sin\theta}. \quad . \quad . \quad . \quad . \quad (E')$$

31. Since $\cos\theta = \sin(\frac{1}{2}\pi + \theta)$, we derive from formula (E),

$$\int \frac{d\theta}{\cos\theta} = \int \frac{d\theta}{\sin(\frac{1}{2}\pi+\theta)} = \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]. \quad . \quad . \quad (F)$$

By employing a process similar to that used in deriving formula (E'), we have also

$$\int \frac{d\theta}{\cos\theta} = \log\frac{1+\sin\theta}{\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad (F')$$

Miscellaneous Trigonometric Integrals.

32. A trigonometric integral may sometimes be reduced, by means of the formulæ for trigonometric transformation, to one of the forms integrated in the preceding articles. For example, let us take the integral

$$\int \frac{d\theta}{a\sin\theta + b\cos\theta}.$$

Putting $a = k\cos\alpha, \qquad b = k\sin\alpha, \quad . \quad . \quad . \quad . \quad (1)$

we have

$$\int \frac{d\theta}{a\sin\theta + b\cos\theta} = \frac{1}{k}\int \frac{d\theta}{\sin(\theta+\alpha)}.$$

Hence by formula (E)

$$\int \frac{d\theta}{a\sin\theta + b\cos\theta} = \frac{1}{k}\log\tan\frac{1}{2}(\theta+\alpha);$$

or, since equations (1) give

$$k = \sqrt{(a^2+b^2)}, \qquad \tan\alpha = \frac{b}{a},$$

$$\int \frac{d\theta}{a\sin\theta + b\cos\theta} = \frac{1}{\sqrt{(a^2+b^2)}}\log\tan\frac{1}{2}\left[\theta + \tan^{-1}\frac{b}{a}\right].$$

33. The expression $\sin m\theta \sin n\theta \, d\theta$ may be integrated by means of the formula

$$\cos(m-n)\theta - \cos(m+n)\theta = 2\sin m\theta \sin n\theta;$$

whence

$$\int \sin m\theta \sin n\theta \, d\theta = \frac{\sin(m-n)\theta}{2(m-n)} - \frac{\sin(m+n)\theta}{2(m+n)}. \quad . \quad (1)$$

In like manner, from

$$\cos(m-n)\theta + \cos(m+n)\theta = 2\cos m\theta \cos n\theta,$$

we derive

$$\int \cos m\theta \cos n\theta \, d\theta = \frac{\sin(m-n)\theta}{2(m-n)} + \frac{\sin(m+n)\theta}{2(m+n)}. \quad . \quad (2)$$

When $m = n$, the first term of the second member of each of these equations takes an indeterminate form. Evaluating this term, we have

$$\int \sin^2 n\theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2n\theta}{4n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\int \cos^2 n\theta \, d\theta = \frac{\theta}{2} + \frac{\sin 2n\theta}{4n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Using the limits 0 and π we have, from (1) and (2), *when* m *and* n *are unequal integers,*

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0; \quad . \quad . \quad (5)$$

but, when m and n are *equal integers*, we have from (3) and (4)

$$\int_0^\pi \sin^2 n\theta \, d\theta = \int_0^\pi \cos^2 n\theta \, d\theta = \frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

34. To integrate $\sqrt{(1+\cos\theta)}\, d\theta$, we use the formula

$$2\cos^2 \tfrac{1}{2}\theta = 1 + \cos\theta.$$

whence $$\sqrt{(1 + \cos\theta)} = \pm \sqrt{2} \cos \tfrac{1}{2}\theta,$$

in which the positive sign is to be taken, provided the value of θ is between 0 and π. Supposing this to be the case, we have

$$\int \sqrt{(1 + \cos\theta)}\, d\theta = \sqrt{2} \int \cos \tfrac{1}{2}\theta\, d\theta$$

$$= 2\sqrt{2} \sin \tfrac{1}{2}\theta.$$

For example, we have the definite integral

$$\int_0^{\frac{\pi}{2}} \sqrt{(1 + \cos\theta)}\, d\theta = 2\sqrt{2} \sin \frac{\pi}{4} = 2.$$

Integration of $\dfrac{d\theta}{a + b\cos\theta}$.

35. By means of the formulæ.

$$1 = \cos^2 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta \quad \text{and} \quad \cos\theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta,$$

we have

$$\int \frac{d\theta}{a + b\cos\theta} = \int \frac{d\theta}{(a + b)\cos^2 \tfrac{1}{2}\theta + (a - b)\sin^2 \tfrac{1}{2}\theta}.$$

Multiplying numerator and denominator by $\sec^2 \tfrac{1}{2}\theta$, this becomes

$$\int \frac{\sec^2 \tfrac{1}{2}\theta\, d\theta}{a + b + (a - b)\tan^2 \tfrac{1}{2}\theta},$$

and, putting for abbreviation

$$\tan \tfrac{1}{2}\theta = y,$$

we have, since $\tfrac{1}{2}\sec^2 \tfrac{1}{2}\theta\, d\theta = dy$,

$$\int \frac{d\theta}{a + b\cos\theta} = 2 \int \frac{dy}{a + b + (a - b)y^2}.$$

The form of this integral depends upon the relative values of a and b. Assuming a to be positive, if b, which may be either positive or negative, is numerically less than a, we may put

$$\frac{a+b}{a-b} = c^2.$$

The integral may then be written in the form

$$\frac{2}{a-b}\int \frac{dy}{c^2+y^2},$$

the value of which is, by formula (k'),

$$\frac{2}{c(a-b)}\tan^{-1}\frac{y}{c}.$$

Hence, substituting their values for y and c, we have, in this case,

$$\int \frac{d\theta}{a+b\cos\theta} = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\tfrac{1}{2}\theta\right]. \quad (G)$$

If, on the other hand, b is numerically greater than a, this expression for the integral involves imaginary quantities; but putting

$$\frac{b+a}{b-a} = c^2,$$

the integral becomes

$$\frac{2}{b-a}\int \frac{dy}{c^2-y^2},$$

the value of which is, by formula (A'), Art. 17,

$$\frac{1}{c(b-a)}\log\frac{c+y}{c-y}.$$

Therefore, in this case,

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{(b+a)} + \sqrt{(b-a)}\tan\frac{1}{2}\theta}{\sqrt{(b+a)} - \sqrt{(b-a)}\tan\frac{1}{2}\theta}. \quad (G')$$

36. If $e < 1$, formula (G) of the preceding article gives

$$\int \frac{d\theta}{1 + e\cos\theta} = \frac{2}{\sqrt{(1 - e^2)}} \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}\theta\right]. \quad (1)$$

Putting

$$\sqrt{\frac{1-e}{1+e}} \cdot \tan\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\phi, \quad (2)$$

and noticing that $\phi = 0$ when $\theta = 0$, we may write

$$\int_0 \frac{d\theta}{1 + e\cos\theta} = \frac{\phi}{\sqrt{(1 - e^2)}}. \quad (3)$$

Now, if in equation (1) we put ϕ for θ and change the sign of e, we obtain

$$\int_0 \frac{d\phi}{1 - e\cos\phi} = \frac{2}{\sqrt{(1 - e^2)}} \tan^{-1}\left[\sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}\phi\right];$$

hence, by equation (2),

$$\int_0 \frac{d\phi}{1 - e\cos\phi} = \frac{\theta}{\sqrt{(1 - e^2)}}. \quad (4)$$

Equations (3) and (4) are equivalent to

$$\frac{d\theta}{1 + e\cos\theta} = \frac{d\phi}{\sqrt{(1 - e^2)}}, \quad (5)$$

and

$$\frac{d\phi}{1 - e\cos\phi} = \frac{d\theta}{\sqrt{(1 - e^2)}}, \quad (6)$$

the product of which gives

$$(1 + e\cos\theta)(1 - e\cos\phi) = 1 - e^2 \quad . \quad . \quad . \quad . \quad (7)$$

By means of these relations any expression of the form

$$\int \frac{d\theta}{(1 + e\cos\theta)^n},$$

where n is a positive integer, may be reduced to an integrable form. For

$$\int \frac{d\theta}{(1 + e\cos\theta)^n} = \int \frac{d\theta}{1 + e\cos\theta} \frac{1}{(1 + e\cos\theta)^{n-1}};$$

hence, by equations (5) and (7),

$$\int_0 \frac{d\theta}{(1 + e\cos\theta)^n} = \frac{1}{(1 - e^2)^{n-\frac{1}{2}}} \int_0 (1 - e\cos\phi)^{n-1}\, d\phi.$$

By expanding $(1 - e\cos\phi)^{n-1}$, the last expression is reduced to a series of integrals involving powers of $\cos\phi$; these may be evaluated by the methods given in this section and Section VI, and the results expressed in terms of θ by means of equation (2) or of equation (7).

Examples III.

1. $\int \tan^4 mx\, dx,$ $\qquad \frac{\tan^3 mx}{3m} - \frac{\tan mx}{m} + x.$

2. $\int_0^{\frac{\pi}{4}} \tan^7 x\, dx,$ $\qquad \frac{5}{12} - \frac{1}{2}\log 2.$

3. $\int \sec^4(\theta + \alpha)\, d\theta,$ $\qquad \frac{\tan^3(\theta + \alpha)}{3} + \tan(\theta + \alpha).$

4. $\int_0^{\frac{\pi}{m}} \sin^3 mx\, dx$, $\frac{4}{3m}$.

5. $\int \sin^2\theta \cos^3\theta\, d\theta$, $\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}$.

6. $\int \sqrt{(\sin\theta)}\, \cos^5\theta\, d\theta$, $\frac{2}{3}\sin^{\frac{3}{2}}\theta - \frac{4}{7}\sin^{\frac{7}{2}}\theta + \frac{2}{11}\sin^{\frac{11}{2}}\theta$.

7. $\int_0^{\frac{\pi}{2}} \cos^4\theta \sin^3\theta\, d\theta$, $\frac{2}{35}$.

8. $\int \frac{\sin^3\theta\, d\theta}{\sqrt{(\cos\theta)}}$, $\frac{2}{5}\cos^{\frac{5}{2}}\theta - 2\cos^{\frac{1}{2}}\theta$.

9. $\int \frac{d\theta}{\sin^2\theta\cos^2\theta}$, *Multiply by* $\sin^2\theta + \cos^2\theta$. $\tan\theta - \cot\theta$.

10. $\int \frac{\sin^3 x}{\cos^5 x}\, dx$, *See Art.* 28. $\frac{\tan^4 x}{4}$.

11. $\int \frac{d\theta}{\sin^3\theta\cos^3\theta}$, $\frac{1}{2}(\tan^2\theta - \cot^2\theta) + 2\log\tan\theta$.

12. $\int \frac{\sqrt{(\sin\theta)}\, d\theta}{\cos^{\frac{5}{2}}\theta}$, $\frac{2}{3}\tan^{\frac{3}{2}}\theta$.

13. $\int \frac{\sin^3 x\, dx}{\cos^6 x}$, $\frac{1}{5\cos^5 x} - \frac{1}{3\cos^3 x}$.

14. $\int \frac{\sin^2 x\, dx}{\cos^6 x}$, $\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3}$.

15. $\int \sin^2\theta\cos^2\theta\, d\theta$, $\frac{1}{16}[2\theta - \sin 2\theta\cos 2\theta]$.

16. $\int_0^{\frac{\pi}{m}} \sin^2 mx\, dx$, $\qquad \frac{\pi}{2m}$.

17. $\int \frac{\sin^2\theta\, d\theta}{\cos\theta}$, $\qquad \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]-\sin\theta$.

18. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2\theta\, d\theta}{\sin\theta}$, $\qquad \frac{1}{2}(\log 3 - 1)$.

19. $\int \frac{d\theta}{\sin\theta+\cos\theta}$, $\qquad \frac{1}{\sqrt{2}}\log\tan\left[\frac{\theta}{2}+\frac{\pi}{8}\right]$.

20. $\int \frac{dx}{1+\cos x}$, $\qquad \tan\frac{1}{2}x$.

21. $\int_{\frac{\pi}{2}} \frac{dx}{1-\cos x}$, $\qquad 1-\cot\frac{1}{2}x$.

22. $\int \frac{dx}{1\pm\sin x}$,

Multiply both terms of the fraction by $1\mp\sin x$. $\qquad \tan x \mp \sec x$.

23. $\int \frac{d\theta}{\sec\theta\pm\tan\theta}$, $\qquad \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]\pm\log\cos\theta$.

24. $\int \cos\theta\cos 3\theta\, d\theta$. *See Art.* 33. $\qquad \frac{1}{8}\sin 4\theta+\frac{1}{4}\sin 2\theta$.

25. $\int_0^{\frac{\pi}{2}} \cos\theta\cos 2\theta\, d\theta$, $\qquad \frac{1}{3}$.

26. $\int_0^{\frac{\pi}{4}} \sin^2\theta\sin 2\theta\, d\theta$, $\qquad \frac{1}{2}\sin^4\theta\Big]_0^{\frac{\pi}{4}} = \frac{1}{8}$.

27. $\int_0^{\frac{\pi}{2}} \sin 3\theta\sin 2\theta\, d\theta$, $\qquad \frac{2}{5}$.

28. $\int_0 \sin m\theta \cos n\theta \, d\theta$,

$$\frac{1 - \cos(m+n)\theta}{2(m+n)} + \frac{1 - \cos(m-n)\theta}{2(m-n)}.$$

29. $\int \cos x \cos 2x \cos 3x \, dx$,

Reduce products to sums by means of equation (2), *Art.* 33.

$$\frac{1}{4}\left[\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + x\right].$$

30. $\int_0^{\pi} \sqrt{(1 - \cos x)} \, dx$, $2\sqrt{2}$.

31. $\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$, $\frac{1}{ab} \tan^{-1}\left[\frac{b}{a} \tan x\right]$.

32. $\int \frac{dx}{1 + \cos^2 x}$, $\frac{1}{\sqrt{2}} \tan^{-1} \frac{\tan x}{\sqrt{2}}$.

33. $\int \frac{dx}{a^2 \cos^2 x - b^2 \sin^2 x}$, $\frac{1}{2ab} \log \frac{a + b \tan x}{a - b \tan x}$.

34. $\int \frac{\sin x \, dx}{\sqrt{(3 \cos^2 x + 4 \sin^2 x)}}$, $\cos^{-1}\{\frac{1}{2} \cos x\}$.

35. $\int \frac{\sin x \cos^2 x \, dx}{1 + a^2 \cos^2 x}$,

Putting y *for* $\cos x$, *the integral becomes* $-\int \frac{y^2 \, dy}{1 + a^2 y^2}$.

$$-\frac{\cos x}{a^2} + \frac{\tan^{-1}(a \cos x)}{a^3}.$$

36. $\int \frac{d\theta}{a + b \sin \theta}$.

Put $\sin \theta = \cos(\theta - \frac{1}{2}\pi)$, *and use formulas* (G) *and* (G').

If $a > b$, $\frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan \frac{2\theta - \pi}{4}\right]$.

If $a < b$, $\frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{(b+a)} + \sqrt{(b-a)} \tan(\frac{1}{2}\theta - \frac{1}{4}\pi)}{\sqrt{(b+a)} - \sqrt{(b-a)} \tan(\frac{1}{2}\theta - \frac{1}{4}\pi)}$.

37. $\int \frac{d\theta}{3 + 5\cos\theta}$, $\quad \frac{1}{4} \log \frac{2 + \tan \frac{1}{2}\theta}{2 - \tan \frac{1}{2}\theta}$.

38. $\int \frac{d\theta}{5 + 3\cos\theta}$, $\quad \frac{1}{2} \tan^{-1}[\frac{1}{2} \tan \frac{1}{2}\theta]$.

39. $\int \frac{d\theta}{5 - 4\cos\theta}$, $\quad \frac{2}{3} \tan^{-1}\{3 \tan \frac{1}{2}\theta\}$.

40. $\int \frac{d\theta}{2\cos\theta - 1}$, $\quad \frac{1}{\sqrt{3}} \log \frac{1 + \sqrt{3} \tan \frac{1}{2}\theta}{1 - \sqrt{3} \tan \frac{1}{2}\theta}$.

41. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3 - \cos\theta}$, $\quad \frac{\tan^{-1}\sqrt{2}}{\sqrt{2}}$.

42. $\int_0^{\frac{\pi}{3}} \frac{d\theta}{2 - \cos\theta}$, $\quad \frac{\pi}{2\sqrt{3}}$.

43. $\int \frac{d\theta}{(1 + e\cos\theta)^2}$, *See Art.* 36.

$$\frac{1}{(1-e^2)^{\frac{3}{2}}} \cos^{-1} \frac{e + \cos\theta}{1 + e\cos\theta} - \frac{e}{1-e^2} \frac{\sin\theta}{1 + e\cos\theta}.$$

44. $\int_0^{\pi} \frac{d\theta}{(1 + e\cos\theta)^3}$, $\quad \frac{(2 + e^2)\pi}{2(1-e^2)^{\frac{5}{2}}}$.

45. $\int \frac{p\cos x + q\sin x}{a\cos x + b\sin x}\,dx,$

Solution :—

By adding and subtracting an undetermined constant, the fraction may be written in the form

$$\frac{p\cos x + q\sin x + A(a\cos x + b\sin x)}{a\cos x + b\sin x} - A,$$

we may now assume

$$p\cos x + q\sin x + A(a\cos x + b\sin x) = k(b\cos x - a\sin x);$$

the expression is then readily integrated, and A and k so determined as to make the equation last written an identity. The result is

$$\int \frac{p\cos x + q\sin x}{a\cos x + b\sin x}\,dx = \frac{ap + bq}{a^2 + b^2}x + \frac{bp - aq}{a^2 + b^2}\log(a\cos x + b\sin x).$$

46. $\int \frac{dx}{a + b\tan x}$, *See Ex.* 45.

$$\frac{ax}{a^2 + b^2} + \frac{b}{a^2 + b^2}\log(a\cos x + b\sin x).$$

47. Find the area of the ellipse

$$x = a\cos\phi \qquad y = b\sin\phi.$$

$$-4ab\int_{\frac{1}{2}\pi}^{0} \sin^2\phi\,d\phi = \pi ab.$$

48. Find the area of the cycloid

$$x = a(\psi - \sin\psi) \qquad y = a(1 - \cos\psi).$$

$$a^2\int_0^{2\pi} (1 - \cos\psi)^2\,d\psi = 3a^2\pi.$$

49. Find the area of the trochoid $(b < a)$

$$x = a\psi - b\sin\psi \qquad y = a - b\cos\psi.$$

$$(2a^2 + b^2)\,\pi.$$

50. Find the area of the loop, and also the area between the curve and the asymptote, in the case of the strophoid whose polar equation is

$$r = a\,(\sec\theta \pm \tan\theta).$$

Solution :—

Using θ as an auxiliary variable, we have

$$x = a\,(1 \pm \sin\theta) \qquad y = a\left[\tan\theta \pm \frac{\sin^2\theta}{\cos\theta}\right],$$

the upper sign corresponding to the infinite branch, and the lower to the loop. Hence, for the half areas we obtain

$$+ a^2\int_0^{\frac{1}{2}\pi} \sin\theta\,d\theta + a^2\int_0^{\frac{1}{2}\pi} \sin^2\theta\,d\theta = a^2\left[1 + \frac{\pi}{4}\right]$$

and

$$-a^2\int_{\frac{1}{2}\pi}^{0} \sin\theta\,d\theta + a^2\int_{\frac{1}{2}\pi}^{0} \sin^2\theta\,d\theta = a^2\left[1 - \frac{\pi}{4}\right].$$

CHAPTER II.

METHODS OF INTEGRATION—CONTINUED.

IV.

Integration by Change of Independent Variable.

37. IF x is the independent variable used in expressing an integral, and y is any function of x, the integral may be expressed in terms of y, by substituting for x and dx their values in terms of y and dy. By properly assuming the function y, the integral may frequently be made to take a directly integrable form. For example, the integral

$$\int \frac{x\,dx}{(ax+b)^2}$$

will obviously be simplified by assuming

$$y = ax + b$$

for the new independent variable. This assumption gives

$$x = \frac{y-b}{a}, \qquad \text{whence} \qquad dx = \frac{dy}{a};$$

substituting, we have

$$\int \frac{x\,dx}{(ax+b)^2} = \frac{1}{a^2}\int \frac{(y-b)\,dy}{y^2}$$

$$= \frac{1}{a^2}\log y + \frac{b}{a^2 y};$$

or replacing y by x in the result,

$$\int \frac{x\,dx}{(ax+b)^2} = \frac{1}{a^2}\log(ax+b) + \frac{b}{a^2(ax+b)}.$$

38. Again, if in the integral

$$\int \frac{dx}{e^x - 1}$$

we put $y = e^x$, whence

$$x = \log y, \qquad \text{and} \qquad dx = \frac{dy}{y},$$

we have

$$\int \frac{dx}{e^x - 1} = \int \frac{dy}{y(y-1)}.$$

Hence, by formula (A), Art. 17,

$$\int \frac{dx}{e^x - 1} = \log \frac{y-1}{y} = \log(e^x - 1) - x.$$

It is easily seen that, by this change of independent variable, any integral in which the coefficient of dx is a rational function of e^x, may be transformed into one in which the coefficient of dy is a rational function of y.

Transformation of Trigonometric Forms.

39. When in a trigonometric integral the coefficient of $d\theta$ is a rational function of $\tan\theta$, the integral will take a rational algebraic form if we put

$$\tan\theta = x, \qquad \text{whence} \qquad d\theta = \frac{dx}{1+x^2}.$$

For example, by this transformation, we have

$$\int \frac{d\theta}{1 + \tan\theta} = \int \frac{dx}{(1 + x^2)(1 + x)}.$$

Decomposing the fraction in the latter integral, we have

$$\int \frac{d\theta}{1 + \tan\theta} = \frac{1}{2}\int \frac{dx}{1 + x^2} - \frac{1}{2}\int \frac{x\,dx}{1 + x^2} + \frac{1}{2}\int \frac{dx}{1 + x}$$

$$= \tfrac{1}{2}\tan^{-1}x - \tfrac{1}{4}\log(1 + x^2) + \tfrac{1}{2}\log(1 + x)$$

$$= \frac{1}{2}\left[\theta + \log\frac{1 + \tan\theta}{\sec\theta}\right],$$

or $$\int \frac{d\theta}{1 + \tan\theta} = \tfrac{1}{2}[\theta + \log(\cos\theta + \sin\theta)].$$

40. The method given in the preceding article may be employed when the coefficient of $d\theta$ is a *homogeneous rational function of* $\sin\theta$ and $\cos\theta$, *of a degree indicated by an even integer;* for such a function is a rational function of $\tan\theta$. It may also be noticed that, when the coefficient of $d\theta$ is *any rational function* of $\sin\theta$ and $\cos\theta$, the integral becomes rational and algebraic if we put

$$z = \tan\frac{\theta}{2};$$

for this gives

$$\sin\theta = \frac{2z}{1 + z^2}, \qquad \cos\theta = \frac{1 - z^2}{1 + z^2}, \qquad d\theta = \frac{2dz}{1 + z^2}.$$

This transformation has in fact been already employed in the integration of $\dfrac{d\theta}{a + b\cos\theta}$. See Art. 35.

Limits of the Transformed Integral.

41. When a definite integral is transformed by a change of independent variable, it is necessary to make a corresponding change in the limits. If, for example, in the integral

$$\int_a^\infty \frac{dx}{(a^2 + x^2)^2}$$

we put $x = a \tan \theta$, whence $dx = a \sec^2 \theta \, d\theta$,

we must at the same time replace the limits a and ∞, which are values of x, by $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$, the corresponding values of θ. Thus

$$\int_a^\infty \frac{dx}{(a^2 + x^2)^2} = \frac{1}{a^3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

$$= \frac{1}{2a^3}\Big[\theta + \sin \theta \cos \theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi - 2}{8a^3}.$$

The Reciprocal of x taken as the New Independent Variable.

42. In the case of fractional integrals, it is sometimes useful to take the reciprocal of x as the new independent variable. For example, let the given integral be

$$\int \frac{dx}{x^3(x + 1)^2}.$$

Putting $x = \frac{1}{y}$, whence $dx = -\frac{dy}{y^2}$,

we have

$$\int \frac{dx}{x^3(x+1)^2} = -\int \frac{y^3\,dy}{y^2\left(1+\frac{1}{y}\right)^2} = -\int \frac{y^3\,dy}{(y+1)^2}.$$

Transforming again by putting $z = y + 1$, the integral becomes

$$-\int \frac{(z-1)^3}{z^2}\,dz = -\int z\,dz + 3\int dz - 3\int \frac{dz}{z} + \int \frac{dz}{z^2}$$

$$= -\frac{z^2}{2} + 3z - 3\log z - \frac{1}{z}.$$

Therefore, since $z = y + 1 = \frac{1}{x} + 1 = \frac{x+1}{x}$,

$$\int \frac{dx}{x^3(x+1)^2} = -\frac{1}{2x^2} + \frac{2}{x} + \frac{1}{x+1} - 3\log\frac{x+1}{x} + C.$$

43. In the above example we see that the single substitution $z = \frac{x+1}{x}$ is equivalent to the two substitutions which are suggested in the process. In like manner, in the case of the more general integral

$$\int \frac{dx}{(x-a)^m(x-b)^n},$$

the successive transformations which suggest themselves are found to be equivalent to a single one in which the new variable is

$$z = \frac{x-a}{x-b}, \quad \text{whence} \quad x = \frac{bz-a}{z-1}.$$

The result is

$$\frac{1}{(a-b)^{m+n-1}}\int\frac{(1-z)^{m+n-2}}{z^m}dz,$$

which, when $m+n-2$ is a positive integer, is directly integrable after expansion by the Binomial Theorem.

A Power of x taken as the New Independent Variable.

44. The transformation of an integral by the assumption

$$y = x^n \quad \ldots\ldots\ldots \quad (1)$$

is not generally useful, since the substitution

$$x = y^{\frac{1}{n}}, \qquad \text{whence} \qquad dx = \frac{1}{n}y^{\frac{1}{n}-1}dy,$$

will usually introduce radicals. Exceptional cases, however, occur. For, since logarithmic differentiation of equation (1) gives

$$\frac{dx}{x} = \frac{dy}{ny}, \quad \ldots\ldots\ldots \quad (2)$$

it is evident that, *if the expression to be integrated is the product of* $\frac{dx}{x}$ *and a function of* x^n, *the transformed expression will be the product of* $\frac{dy}{ny}$ *and the like function of* y.

For example, the substitution $y = x^4$ transforms

$$\frac{(x^4-1)dx}{x(x^4+1)} \quad \text{into} \quad \frac{(y-1)dy}{4y(y+1)}.$$

Hence, decomposing the fraction in the latter expression,

$$\int \frac{(x^4 - 1)dx}{x(x^4 + 1)} = \frac{1}{4} \log \frac{(y + 1)^2}{y} = \log \frac{\sqrt{(x^4 + 1)}}{x}.$$

Again, putting $x^3 = y^2$ or $y = x^{\frac{3}{2}}$, we have

$$\int \frac{dx}{x\sqrt{(x^3 - a^3)}} = \frac{2}{3}\int \frac{dy}{y\sqrt{(y^2 - a^3)}} = \frac{2}{3a^{\frac{3}{2}}} \sec^{-1} \frac{y}{a^{\frac{3}{2}}} = \frac{2}{3a^{\frac{3}{2}}} \sec^{-1} \left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

45. When this method is applied to an integral whose form at the same time suggests the employment of the reciprocal, as in Art. 42, we may at once assume $y = x^{-n}$. Thus, given the integral

$$\int_1^\infty \frac{dx}{x^4(2 + x^3)};$$

putting $y = x^{-3}$, whence $\frac{dx}{x} = -\frac{dy}{3y}$,

we obtain

$$\int_1^\infty \frac{dx}{x^4(2 + x^3)} = -\frac{1}{3}\int_1^0 \frac{y\,dy}{2y + 1}$$

$$= -\frac{y}{6} + \frac{\log(2y + 1)}{12}\Big]_1^0 = \frac{2 - \log 3}{12}.$$

Examples IV.

1. $\int \frac{1 + x}{(2 + x)^2} dx$, $\log(2 + x) + \frac{1}{2 + x}$

2. $\int \frac{x\,dx}{(1-x)^3}$, $\qquad \frac{2x-1}{2(1-x)^2}$.

3. $\int \frac{x^2-x+1}{(2x+1)^2}\,dx$, $\qquad \frac{2x+1}{8} - \frac{\log(2x+1)}{2} - \frac{7}{8(2x+1)}$.

4. $\int_{-1}^{0} \frac{x^2\,dx}{(x+2)^3}$, $\qquad \log y + \frac{4y-2}{y^2}\Big]_1^2 = \log 2 - \frac{1}{2}$.

5. $\int \frac{dx}{1+e^x}$ $\qquad x - \log(1+e^x)$.

6. $\int \frac{dx}{e^x - e^{-x}}$, $\qquad \frac{1}{2}\log\frac{e^x-1}{e^x+1}$.

7. $\int_{-\infty}^{0} \frac{e^{2x}\,dx}{e^x+1}$, $\qquad 1 - \log 2$.

8. $\int \frac{e^x+1}{1-e^{-x}}\,dx$, $\qquad e^x + 2\log(e^x-1)$.

9. $\int \frac{2+\tan\theta}{3-\tan\theta}\,d\theta$, $\qquad \frac{\theta - \log(3\cos\theta - \sin\theta)}{2}$.

10. $\int \frac{d\theta}{\tan^2\theta - 1}$, $\qquad \frac{1}{4}\log\frac{\tan\theta-1}{\tan\theta+1} - \frac{\theta}{2}$.

11. $\int \frac{\tan^2\theta\,d\theta}{\tan^2\theta - 1}$, $\qquad \frac{1}{4}\log\frac{\tan\theta-1}{\tan\theta+1} + \frac{\theta}{2}$.

12. $\int \frac{\cos\theta\,d\theta}{a\cos\theta - b\sin\theta}$, $\qquad \frac{a\theta - b\log(a\cos\theta - b\sin\theta)}{a^2+b^2}$.

13. $\int \frac{\cos\theta\, d\theta}{\cos(\alpha+\theta)}$, *Put* $\theta' = \alpha + \theta$.

$$(\theta+\alpha)\cos\alpha - \sin\alpha \log\cos(\theta+\alpha).$$

14. $\int \frac{\sin(\theta+\alpha)}{\sin(\theta+\beta)}\, d\theta$,

$$(\theta+\beta)\cos(\alpha-\beta) + \sin(\alpha-\beta)\log\sin(\theta+\beta).$$

15. $\int \tan(\theta+\alpha)\cos\theta\, d\theta$, $\quad -\cos\theta + \sin\alpha \log\tan\frac{2\theta+2\alpha+\pi}{4}$

16. $\int_0^a \frac{\cos\theta\, d\theta}{\sin(\alpha+\theta)}$, $\qquad \cos\alpha\log(2\cos\alpha) + \alpha\sin\alpha.$

17. $\int_0^{\frac{\pi}{3}} \frac{\cos\frac{1}{2}\theta}{\cos\theta}\, d\theta$, $\qquad \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+2\sin\theta'}{\sqrt{2}-2\sin\theta'}\Big]_0^{\frac{\pi}{6}} = \frac{\log(3+2\sqrt{2})}{\sqrt{2}}$.

18. $\int \frac{\sin\frac{1}{2}\theta\, d\theta}{\sin\theta}$, $\qquad \log\tan\frac{\pi+\theta}{4}$.

19. $\int \frac{x^3\, dx}{(a^2+x^2)^2}$, $\qquad \frac{1}{2}\log(a^2+x^2) + \frac{a^2}{2(a^2+x^2)}$.

20. $\int \frac{dx}{x^3(1+x^2)}$, $\qquad \log\frac{\sqrt{(1+x^2)}}{x} - \frac{1}{2x^2}$.

21. $\int_0^\infty \frac{x^2\, dx}{(1+x^2)^3}$, $\qquad \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta = \frac{\pi}{16}$.

22. $\int_1^\infty \frac{dx}{x^3(1+x^2)^2}$, $\qquad \frac{3}{4} - \log 2$.

23. $\int \frac{dx}{x^2(x+1)}$, $\quad -\frac{1}{2x^2} + \frac{1}{x} - \log\frac{x+1}{x}$.

24. $\int \frac{dx}{(1-x)^2 x}$, $\quad \frac{1}{2(1-x)^2} + \frac{1}{1-x} + \log\frac{x}{1-x}$.

25. $\int_1^\infty \frac{dx}{x^3(x^2+2)}$, $\quad -\frac{y^2}{4} + \frac{1}{8}\log(2y^2+1)\Big]_1^\infty = \frac{2-\log 3}{8}$.

26. $\int \frac{dx}{x(a+bx^4)}$, $\quad \frac{1}{4a}\log\frac{x^4}{a+bx^4}$.

27. $\int \frac{dx}{x(x^n+a^n)}$, $\quad \frac{1}{na^n}\log\frac{x^n}{x^n+a^n}$.

28. $\int \frac{(x^3+1)\,dx}{x(x^3-1)}$, $\quad \frac{2}{3}\log(x^3-1) - \log x$.

29. $\int \frac{dx}{x\sqrt{(x^n-a^n)}}$, $\quad \frac{2}{na^{\frac{n}{2}}}\sec^{-1}\left(\frac{x}{a}\right)^{\frac{n}{2}}$.

V.

Integrals Containing Radicals.

46. An integral containing a single radical, in which the expression under the radical sign is of the first degree, is *rationalized*, that is, transformed into a rational integral, by taking the radical as the value of the new independent variable. Thus, given the integral

$$\int \frac{dx}{1+\sqrt{(x+1)}},$$

putting $y = \sqrt{(x+1)}$,

whence $x = y^2 - 1$, and $dx = 2y\,dy$,

we have

$$\int \frac{dx}{1+\sqrt{(x+1)}} = 2\int \frac{y\,dy}{1+y} = 2\int dy - 2\int \frac{dy}{1+y}$$

$$= 2y - 2\log(1+y)$$

$$= 2\sqrt{(x+1)} - 2\log[1+\sqrt{(x+1)}].$$

47. The same method evidently applies whenever all the radicals which occur in the integral are powers of a single radical, in which the expression under the radical sign is linear. Thus, in the integral

$$\int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}} + (x-1)^{\frac{1}{2}}},$$

the radicals are powers of $(x-1)^{\frac{1}{6}}$; hence we put $y = (x-1)^{\frac{1}{6}}$, and obtain

$$\int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}} + (x-1)^{\frac{1}{2}}} = 6\int_0^1 \frac{y^5\,dy}{y^4 + y^3}$$

$$= 6\int_0^1 (y-1)\,dy + 6\int_0^1 \frac{dy}{y+1} = -3 + 6\log 2.$$

48. An integral in which a binomial expression occurs under the radical sign can sometimes be reduced to the form considered above by the method of Art. 44. For example, since

$$\int \frac{dx}{x(x^3+1)^{\frac{1}{2}}}$$

fulfils the condition given in Art. 44 when $n = 3$, the quantity under the radical sign may be reduced to the first degree. Hence, in accordance with Art. 46, we may take the radical as the value of the new independent variable. Thus, putting

$$z = (x^3 + 1)^{\frac{1}{4}},$$

whence $x^3 = z^4 - 1$, and $\frac{dx}{x} = \frac{4z^3\,dz}{3(z^4 - 1)}$,

we have

$$\int \frac{dx}{x(x^3 + 1)^{\frac{1}{4}}} = \frac{4}{3}\int \frac{z^2\,dz}{z^4 - 1}.$$

Decomposing the fraction in the latter integral (see Art. 22), we have finally

$$\int \frac{dx}{x(x^3 + 1)^{\frac{1}{4}}} = \frac{2}{3}\tan^{-1}\left[(x^3 + 1)^{\frac{1}{4}}\right] + \frac{1}{3}\log\frac{(x^3 + 1)^{\frac{1}{4}} - 1}{(x^3 + 1)^{\frac{1}{4}} + 1}.$$

Radicals of the Form $\sqrt{(ax^2 + b)}$.

49. It is evident that the method given in the preceding article is applicable to all integrals of the general form

$$\int x^{2m+1}(ax^2 + b)^{n+\frac{1}{2}}\,dx, \quad \ldots\ldots \quad (1)$$

in which m and n are positive or negative integers. These integrals are therefore rationalized by putting

$$y = \sqrt{(ax^2 + b)}.$$

Putting $m = 0$, the form (1) includes the directly integrable case

$$\int (ax^2 + b)^{n+\frac{1}{2}} x\,dx.$$

50. As an illustration let us take the integral

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}};$$

putting $$y = \sqrt{(x^2 + a^2)},$$

whence $$x^2 = y^2 - a^2, \qquad \text{and} \qquad \frac{dx}{x} = \frac{y\,dy}{y^2 - a^2},$$

we have

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}} = \int \frac{dy}{y^2 - a^2}.$$

Hence, by equation (A') Art. 17,

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}} = \frac{1}{2a} \log \frac{y - a}{y + a} = \frac{1}{2a} \log \frac{\sqrt{(x^2 + a^2)} - a}{\sqrt{(x^2 + a^2)} + a}.$$

Rationalizing the denominator of the fraction in this result, we have

$$\frac{\sqrt{(x^2 + a^2)} - a}{\sqrt{(x^2 + a^2)} + a} = \frac{[\sqrt{(x^2 + a^2)} - a]^2}{x^2}.$$

Therefore

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}} = \frac{1}{a} \log \frac{\sqrt{(x^2 + a^2)} - a}{x} \quad . \quad . \quad . \quad . \quad (H)$$

In a similar manner we may prove that

$$\int \frac{dx}{x\sqrt{(a^2-x^2)}} = \frac{1}{a}\log\frac{a-\sqrt{(a^2-x^2)}}{x} \quad . \quad . \quad . \quad . \quad (I)$$

51. Integrals of the form

$$\int x^{2m}(ax^2+b)^{n+\frac{1}{2}}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are reducible to the form (1) Art. 49, by first putting $y=\frac{1}{x}$. For example:

$$\int \frac{dx}{(ax^2+b)^{\frac{3}{2}}}$$

is of the form (2); but, putting $x=\frac{1}{y}$, whence

$$\sqrt{(ax^2+b)} = \frac{\sqrt{(a+by^2)}}{y} \qquad \text{and} \qquad dx = -\frac{dy}{y^2},$$

we obtain

$$\int \frac{dx}{(ax^2+b)^{\frac{3}{2}}} = -\int \frac{y\,dy}{(a+by^2)^{\frac{3}{2}}}.$$

The resulting expression is in this case directly integrable. Thus

$$\int \frac{dx}{(ax^2+b)^{\frac{3}{2}}} = \frac{1}{b\sqrt{(a+by^2)}} = \frac{x}{b\sqrt{(ax^2+b)}} \quad . \quad . \quad . \quad (J)$$

Integration of $\frac{dx}{\sqrt{(x^2 \pm a^2)}}$.

52. If we assume a new variable z connected with x by the relation

$$z - x = \sqrt{(x^2 \pm a^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have, by squaring,

$$z^2 - 2zx = \pm a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, by differentiating this equation,

$$2(z - x)\,dz - 2z\,dx = 0;$$

whence

$$\frac{dx}{z - x} = \frac{dz}{z},$$

or by equation (1),

$$\frac{dx}{\sqrt{(x^2 \pm a^2)}} = \frac{dz}{z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating equation (3), we obtain

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log z = \log\,[x + \sqrt{(x^2 \pm a^2)}]. \quad . \quad . \quad (K)$$

53. Since the value of x in terms of z, derived from equation (2) of the preceding article, is rational, it is obvious that this transformation may be employed to rationalize any expression which consists of the product of $\frac{dx}{\sqrt{(x^2 \pm a^2)}}$ and a rational function of x. For example, let us find the value of

$$\int \sqrt{(x^2 \pm a^2)}\,dx,$$

which may be written in the form

$$\int (x^2 \pm a^2) \frac{dx}{\sqrt{(x^2 \pm a^2)}}.$$

By equation (2)

$$x = \frac{z^2 \mp a^2}{2z}, \quad \ldots \ldots \quad (4)$$

whence

$$x^2 \pm a^2 = \frac{(z^2 \pm a^2)^2}{4z^2} \quad \ldots \ldots \quad (5)$$

Therefore, by equations (3) and (5),

$$\int \sqrt{(x^2 \pm a^2)}\, dx = \frac{1}{4} \int \frac{(z^2 \pm a^2)^2}{z^3}\, dz$$

$$= \frac{1}{4} \int z\, dz \pm \frac{a^2}{2} \int \frac{dz}{z} + \frac{a^4}{4} \int \frac{dz}{z^3}$$

$$= \frac{z^4 - a^4}{8z^2} \pm \frac{a^2}{2} \log z.$$

By equations (4) and (5), the first term of the last member is equal to $\frac{1}{2} x \sqrt{(x^2 \pm a^2)}$. Hence

$$\int \sqrt{(x^2 \pm a^2)}\, dx = \frac{x \sqrt{(x^2 \pm a^2)}}{2} \pm \frac{a^2}{2} \log [x + \sqrt{(x^2 \pm a^2)}] \quad (L)$$

Transformation to Trigonometric Forms.

54. Integrals involving either of the radicals

$$\sqrt{(a^2 - x^2)}, \qquad \sqrt{(a^2 + x^2)}, \qquad \text{or} \qquad \sqrt{(x^2 - a^2)}$$

can be transformed into rational trigonometric integrals. The transformation is effected in the first case by putting

$$x = a \sin \theta, \qquad \text{whence} \qquad \sqrt{(a^2 - x^2)} = a \cos \theta;$$

in the second case, by putting

$$x = a \tan \theta, \qquad \text{whence} \qquad \sqrt{(a^2 + x^2)} = a \sec \theta;$$

and in the third case, by putting

$$x = a \sec \theta, \qquad \text{whence} \qquad \sqrt{(x^2 - a^2)} = a \tan \theta.$$

55. As an illustration, let us take the integral

$$\int \sqrt{(a^2 - x^2)}\, dx;$$

putting $x = a \sin \theta$, we have $\sqrt{(a^2 - x^2)} = a \cos \theta$, $dx = a \cos \theta\, d\theta$; hence

$$\int \sqrt{(a^2 - x^2)}\, dx = a^2 \int \cos^2 \theta\, d\theta$$

$$= \frac{a^2 \theta}{2} + \frac{a^2 \sin \theta \cos \theta}{2},$$

by formula (*C*) Art. 29. Replacing θ by x in the result,

$$\int \sqrt{(a^2 - x^2)}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x \sqrt{(a^2 - x^2)}}{2} \quad . \quad . \quad . \quad (M)$$

Regarding the radical as a positive quantity, the value of θ may be restricted to the *primary* value of the symbol $\sin^{-1} \frac{x}{a}$ (see Diff. Calc., Art. 57); that is, as x passes from $-a$ to $+a$, θ passes from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

Radicals of the Form $\sqrt{(ax^2 + bx + c)}$.

56. When a radical of the form $\sqrt{(ax^2 + bx + c)}$ occurs in an integral, a simple change of independent variable will cause the radical to assume one of the forms considered in the preceding articles. Thus, if the coefficient of x^2 is positive,

$$\sqrt{(ax^2 + bx + c)} = \sqrt{a}\sqrt{\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]},$$

in which, if we put $x + \frac{b}{2a} = y$, the radical takes the form $\sqrt{(y^2 + a^2)}$ or $\sqrt{(y^2 - a^2)}$, according as $4ac - b^2$ is positive or negative. If a is negative, the radical can in like manner be reduced to the form $\sqrt{(a^2 - y^2)}$ or $\sqrt{(-a^2 - y^2)}$; but the latter will never occur, since it is imaginary for all values of y, and therefore imaginary for all values of x.

For example, by this transformation, the integral

$$\int \frac{dx}{(ax^2 + bx + c)^{\frac{3}{2}}}$$

can be reduced at once to the form ($\mathcal{F}$), Art. 51. Thus

$$\int \frac{dx}{(ax^2 + bx + c)^{\frac{3}{2}}} = \int \frac{dx}{\left[a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}\right]^{\frac{3}{2}}}$$

$$= \frac{x + \frac{b}{2a}}{\frac{4ac - b^2}{4a}\sqrt{(ax^2 + bx + c)}} = \frac{4ax + 2b}{(4ac - b^2)\sqrt{(ax^2 + bx + c)}}.$$

57. When the form of the integral suggests a further change of independent variable, we may at once assume the expression for the new variable in the required form. For example, given the integral

$$\int \sqrt{(2ax - x^2)}\, x\, dx;$$

we have $$\sqrt{(2ax - x^2)} = \sqrt{[a^2 - (x - a)^2]}$$

hence (see Art. 54), if we put $x - a = a \sin\theta$, we have

$$\sqrt{(2ax - x^2)} = a\cos\theta,$$

$$x = a(1 + \sin\theta), \qquad dx = a\cos\theta\, d\theta;$$

$$\therefore \int \sqrt{(2ax - x^2)}\, x\, dx = a^3 \int \cos^2\theta (1 + \sin\theta)\, d\theta$$

$$= \frac{a^3}{2}(\theta + \sin\theta\cos\theta) - \frac{a^3}{3}\cos^3\theta$$

$$= \frac{a^3}{2}\sin^{-1}\frac{x-a}{a} + \frac{a}{2}(x-a)\sqrt{(2ax - x^2)} - \frac{1}{3}(2ax - x^2)^{\frac{3}{2}}$$

$$= \frac{a^3}{2}\sin^{-1}\frac{x-a}{a} + \frac{1}{6}\sqrt{(2ax - x^2)}\,[2x^2 - ax - 3a^2].$$

The Integrals

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} \quad \textit{and} \quad \int \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}}.$$

58. An integral of the form $\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$ may by the method of Art. 56, be reduced to the form (K), Art. 52, or to the form (j'), Art. 10, according as a is positive or negative.

But, when the integral appears in one of the above forms (the quadratic under the radical sign admitting of linear factors), another mode of transforming is often convenient.

Assuming $\beta > \alpha$, we may in the first case, since the difference of the factors is the constant $\beta - \alpha$, put

$$\left.\begin{aligned} x - \alpha &= (\beta - \alpha)\sec^2\theta \\ x - \beta &= (\beta - \alpha)\tan^2\theta \end{aligned}\right\},$$

whence

$$dx = 2(\beta - \alpha)\sec^2\theta\tan\theta\, d\theta,$$

and

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} = 2\int \sec\theta\, d\theta = 2\log(\sec\theta + \tan\theta).$$

Substituting, and omitting the constant $\log\sqrt{(\beta - \alpha)}$,

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} = 2\log[\sqrt{(x-\alpha)} + \sqrt{(x-\beta)}]. \quad (N)$$

In the second case, the sum of the factors being $\beta - \alpha$, we may put

$$\left.\begin{aligned} x - \alpha &= (\beta - \alpha)\sin^2\theta \\ \beta - x &= (\beta - \alpha)\cos^2\theta \end{aligned}\right\},$$

whence

$$dx = 2(\beta - \alpha)\sin\theta\cos\theta\, d\theta,$$

and the quantity under the integral sign reduces to $2d\theta$. Therefore

$$\int \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}. \quad . \quad . \quad . \quad (O)$$

The same transformations may of course be employed when other factors occur in connection with these radicals

It can be shown that the values given in formulas (N) and (O) differ only by constants from the results derived by employing the process given in Art. 56.

Examples V.

1. $\int \sqrt{(a-x)} \cdot x\, dx$, $\quad -\frac{2}{15}(a-x)^{\frac{3}{2}}(3x+2a)$.

2. $\int \sqrt{(x+a)} \cdot x^2\, dx$, $\quad \frac{2}{7}(a+x)^{\frac{7}{2}} - \frac{4a}{5}(a+x)^{\frac{5}{2}} + \frac{2a^2}{3}(a+x)^{\frac{3}{2}}$.

3. $\int \frac{x\, dx}{1+\sqrt{x}}$, $\quad \frac{2}{3}x^{\frac{3}{2}} - x + 2\sqrt{x} - 2\log(1+\sqrt{x})$.

4. $\int \frac{x\, dx}{\sqrt{(x+a)}}$, $\quad \frac{2}{3}(x-2a)\sqrt{(x+a)}$.

5. $\int \frac{dx}{\sqrt{x}-1}$, $\quad 2\sqrt{x} + 2\log(1-\sqrt{x})$.

6. $\int_{-a}^{0} (a+x)^{\frac{1}{3}} x\, dx$, $\quad \frac{3v^7}{7} - \frac{3ay^4}{4}\Big]_0^{a^{\frac{1}{3}}} = -\frac{9a^{\frac{7}{3}}}{28}$.

7. $\int \frac{dx}{x\sqrt{(2ax-a^2)}}$, $\quad \frac{2}{a}\tan^{-1}\sqrt{\frac{2x-a}{a}}$.

8. $\int_0^a (a-x)^{\frac{3}{2}} x^2\, dx$, $\quad -\frac{2y^9}{9} + \frac{4ay^7}{7} - \frac{2a^2y^5}{5}\Big]_{\sqrt{a}}^{0} = \frac{16a^{\frac{9}{2}}}{315}$.

9. $\int \frac{dx}{2x^{\frac{1}{2}} - x^{\frac{1}{3}}}$, $\quad x^{\frac{1}{2}} + \frac{3x^{\frac{1}{3}}}{4} + \frac{3x^{\frac{1}{6}}}{4} + \frac{3\log(2x^{\frac{1}{6}}-1)}{8}$.

10. $\int_0^1 (x+1)^{\frac{3}{2}} x\,dx$, $\quad \frac{3y^6}{8} - \frac{3y^5}{5}\Big]_1^{\sqrt[3]{2}} = \frac{3\sqrt[3]{4}}{10} + \frac{9}{40}$.

11. $\int \frac{1-3x}{\sqrt{(1-x)}}\,dx$, $\quad 2(1+x)\sqrt{(1-x)}$.

12. $\int \frac{x\,dx}{x-\sqrt{(x^2-a^2)}}$, $\quad$ *Rationalize the denominator.*

$$\frac{x^3 + (x^2-a^2)^{\frac{3}{2}}}{3a^2}.$$

13. $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}}$, $\quad \frac{2(x+a)^{\frac{3}{2}} - 2(x+b)^{\frac{3}{2}}}{3(a-b)}$

14. $\int \frac{dx}{x\sqrt{(x^4+1)}}$, $\quad \frac{1}{4}\log\frac{\sqrt{(x^4+1)}-1}{\sqrt{(x^4+1)}+1}$.

15. $\int \frac{\sqrt{(x^4+1)}\,dx}{x}$, $\quad \frac{\sqrt{(x^4+1)}}{2} + \frac{1}{4}\log\frac{\sqrt{(x^4+1)}-1}{\sqrt{(x^4+1)}+1}$.

16. $\int \frac{(x^n+1)(x^n-1)^{\frac{3}{2}}}{x}\,dx$,

$$\frac{2}{n}\left[\frac{(x^n-1)^{\frac{5}{2}}}{5} + \frac{(x^n-1)^{\frac{3}{2}}}{3} - (x^n-1)^{\frac{1}{2}} + \tan^{-1}\sqrt{(x^n-1)}\right].$$

17. $\int_0^a \frac{x^5\,dx}{(x^2+a^2)^{\frac{3}{2}}}$, $\quad \frac{y^3}{3} - 2a^2y - \frac{a^4}{y}\Big]_a^{a\sqrt{2}} = \left[\frac{8}{3} - \frac{11\sqrt{2}}{6}\right]a^3$.

18. $\int \frac{\sqrt{(x^2-a^2)}}{x}\,dx \left[= \int \frac{x^2-a^2}{x\sqrt{(x^2-a^2)}}\,dx\right]$,

$$\sqrt{(x^2-a^2)} - a\sec^{-1}\frac{x}{a}.$$

19. $\int \frac{x^2\,dx}{\sqrt{(x^2+a^2)}}$,

$$\left[= \int \frac{x^2+a^2-a^2}{\sqrt{(x^2+a^2)}}\,dx. \quad \textit{See formulæ } (L) \textit{ and } (K).\right]$$

$$\frac{1}{2}x\sqrt{(x^2+a^2)} - \frac{1}{2}a^2 \log\left[x+\sqrt{(x^2+a^2)}\right].$$

20. $\int \frac{\sqrt{(a^2-x^2)}}{x}\,dx$, $\qquad a \log \frac{a-\sqrt{(a^2-x^2)}}{x} + \sqrt{(a^2-x^2)}.$

21. $\int \frac{dx}{x+\sqrt{(x^2+a^2)}}$,

$$\frac{x}{2a^2}\sqrt{(x^2+a^2)} + \frac{1}{2}\log\left[x+\sqrt{(x^2+a^2)}\right] - \frac{x^2}{2a^2}.$$

22. $\int \frac{\sqrt{(x^2+a^2)}}{x^2}\,dx$, $\qquad$ *See Art.* 51.

$$\log\left[\sqrt{(x^2+a^2)}+x\right] - \frac{\sqrt{(x^2+a^2)}}{x}.$$

23. $\int \frac{dx}{\sqrt{(x^2+a^2)}-a}$,

$$\log\left[\sqrt{(x^2+a^2)}+x\right] - \frac{\sqrt{(x^2+a^2)}}{x} - \frac{a}{x}.$$

24. $\int \frac{x\,dx}{\sqrt{(1+x^4)}}$. *See Formula* (K). $\frac{1}{2}\log\left[x^2+\sqrt{(1+x^4)}\right]$.

25. $\int \sqrt{(ax^2+b)}\,dx, \qquad [a>0] \qquad Put\ \sqrt{(ax^2+b)} = z - x\sqrt{a}.$

$$\frac{b}{2\sqrt{a}} \log\left[x\sqrt{a} + \sqrt{(ax^2+b)}\right] + \frac{1}{2}\, x\sqrt{(ax^2+b)}.$$

26. $\int \frac{dx}{(a+x)\sqrt{(x^2+b^2)}},$

$$\frac{1}{\sqrt{(a^2+b^2)}} \log \frac{x + \sqrt{(x^2+b^2)} + a - \sqrt{(a^2+b^2)}}{x + \sqrt{(x^2+b^2)} + a + \sqrt{(a^2+b^2)}}.$$

27. $\int \frac{dx}{x^2\sqrt{(1+x^2)}},$ $\qquad -\frac{\sqrt{(1+x^2)}}{x}.$

28. $\int_{\frac{1}{2}}^{1} \frac{dx}{x^2\sqrt{(1-x^2)}},$ $\qquad -\cot\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3}.$

29. $\int \frac{x^3\,dx}{(x^2-a^2)^{\frac{3}{2}}},$ $\qquad \frac{x^2-2a^2}{\sqrt{(x^2-a^2)}}.$

30. $\int \frac{dx}{(p+qx)\sqrt{(x^2+1)}},$

$$\frac{1}{\sqrt{(p^2+q^2)}} \log \tan \frac{1}{2}\left[\tan^{-1} x + \tan^{-1}\frac{p}{q}\right].$$

31. $\int \frac{dx}{x^3\sqrt{(x^2-1)}},$ $\qquad \frac{\sqrt{(x^2-1)}}{2x^2} + \frac{1}{2}\sec^{-1} x.$

32. $\int_0^a \frac{x^2\,dx}{(x^2+a^2)^{\frac{3}{2}}}$ $\qquad \log\tan\frac{3\pi}{8} - \frac{\sqrt{2}}{2}.$

33. $\int \frac{dx}{x^4 \sqrt{(x^2-1)}}$, $\quad (2x^2+1)\frac{\sqrt{(x^2-1)}}{3x^3}$.

34. $\int \frac{dx}{(a-x)\sqrt{(a^2-x^2)}}$, $\quad \frac{1}{a}\sqrt{\frac{a+x}{a-x}}$.

35. $\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$, $\quad \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{(1-x^2)}}$.

36. $\int_0^a \frac{x^{\frac{5}{2}}\,dx}{\sqrt{(a-x)}}\left[=\int_0^a \frac{x^3\,dx}{\sqrt{(ax-x^2)}}\right]$

$$\left(\frac{1}{2}a\right)^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^3\,d\theta = \frac{5\pi a^3}{16}$$

37. $\int \frac{dx}{x\sqrt{(2ax-x^2)}}$, $\quad -\frac{1}{a}\sqrt{\frac{2a-x}{x}}$.

38. $\int \frac{dx}{x^3\sqrt{(x^4-1)}}$, *Put* $x^2 = z = \sec\theta$. $\quad \frac{\sqrt{(x^4-1)}}{2x^2}$.

39. $\int_0^a \sqrt{(2ax-x^2)}\cdot dx$, $\quad \frac{a^2\pi}{4}$.

40. $\int_0^a \sqrt{(2ax-x^2)}\cdot x\,dx$,

$$a^3 \int_{-\frac{\pi}{2}}^{0} \cos^2\theta\,(1+\sin\theta)\,d\theta = a^3\left[\frac{\pi}{4}-\frac{1}{3}\right].$$

41. $\int_0^a \sqrt{(2ax-x^2)}\cdot x^2\,dx$,

$$a^4 \int_{-\frac{\pi}{2}}^{0} \cos^2\theta\,(1+\sin\theta)^2\,d\theta = a^4\left[\frac{5\pi}{16}-\frac{2}{3}\right].$$

42. $\int \frac{dx}{\sqrt{(2ax + x^2)}}$,

by Art. 56, $\log [x + a + \sqrt{(2ax + x^2)}] + C$;

by Art. 58, $2 \log [\sqrt{x} + \sqrt{(2a + x)}] + C'$.

43. $\int \frac{x\,dx}{\sqrt{(2ax + x^2)}}$, $\sqrt{(2ax + x^2)} - a \log [x + a + \sqrt{(2ax + x^2)}]$.

44. $\int \sqrt{\frac{x}{2a - x}}\, dx \left[= \int \frac{x\,dx}{\sqrt{(2ax - x^2)}} \right]$,

$$a \sin^{-1} \frac{x - a}{a} - \sqrt{(2ax - x^2)}.$$

45. $\int \frac{dx}{\sqrt{(5 + 4x - x^2)}}$, *by Art.* 56, $\sin^{-1} \frac{x - 2}{3} + C$

by Art. 58, $2 \sin^{-1} \sqrt{\frac{x + 1}{6}} + C'$.

46. $\int_0^a \frac{dx}{\sqrt{(ax - x^2)}}$, $2 \sin^{-1} \sqrt{\frac{x}{a}} \Big]_0^a = \pi$.

47. $\int \frac{x^2\,dx}{\sqrt{(3 + 2x - x^2)}}$, $3 \sin^{-1} \frac{x - 1}{2} - \frac{(x + 3)\sqrt{(3 + 2x - x^2)}}{2}$.

48. $\int_{-2}^{-\frac{1}{2}} \frac{dx}{\sqrt{(2 - x - x^2)}}$, $\frac{\pi}{2}$.

49. $\int_a^{2a} \frac{dx}{\sqrt{(x^2 - ax)}}$, $\log (3 + 2\sqrt{2})$.

50. Find the area included by the rectangular hyperbola

$$y^2 = 2ax + x^2,$$

and the double ordinate of the point for which $x = 2a$.

$$a^2[6\sqrt{2} - \log(3 + 2\sqrt{2})].$$

51. Find the area included between the cissoid

$$x(x^2 + y^2) = 2ay^2$$

and the coordinates of the point (a, a); also the whole area between the curve and its asymptote.

$$\left(\frac{3}{4}\pi - 2\right)a^2, \quad \text{and} \quad 3\pi a^2.$$

52. Find the area of the loop of the strophoid

$$x(x^2 + y^2) + a(x^2 - y^2) = 0;$$

also the area between the curve and its asymptote.

$$2a^2\left(1 - \frac{\pi}{4}\right), \quad \text{and} \quad 2a^2\left(1 + \frac{\pi}{4}\right).$$

For the loop put $y = -x\frac{a + x}{\sqrt{(a^2 - x^2)}}$, *since* x *is negative between the limits* $-a$ *and* 0.

53. Show that the area of the segment of an ellipse between the minor axis and any double ordinate is $ab \sin^{-1}\frac{x}{a} + xy$.

VI.

Integration by Parts.

59. Let u and v be any two functions of x; then since

$$d(uv) = u\,dv + v\,du,$$

$$uv = \int u\,dv + \int v\,du,$$

whence
$$\int u\,dv = uv - \int v\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By means of this formula, the integration of an expression of the form $u\,dv$, in which dv is the differential of a known function v, may be made to depend upon the integration of the expression $v\,du$. For example, if

$$u = \cos^{-1}x \qquad \text{and} \qquad dv = dx,$$

we have

$$du = -\frac{dx}{\sqrt{(1-x^2)}};$$

hence, by equation (1),

$$\int \cos^{-1}x \cdot dx = x\cos^{-1}x + \int \frac{x\,dx}{\sqrt{(1-x^2)}},$$

in which the new integral is directly integrable; therefore

$$\int \cos^{-1}x \cdot dx = x\cos^{-1}x - \sqrt{(1-x^2)}.$$

The employment of this formula is called *integration by parts.*

Geometrical Illustration.

60. The formula for integration by parts may be geometrically illustrated as follows. Assuming rectangular axes, let the curve be constructed in which the abscissa and ordinate of each point are corresponding values of v and u, and let this curve cut one of the axes in B. From any point P of this curve draw PR and PS, perpendicular to the axes. Now the area $PBOR$ is a value of the indefinite integral $\int u\,dv$, and in like manner the area PBS is a value of $\int v\,du$;

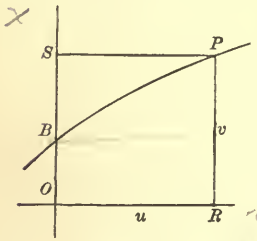

FIG. 2.

and we have

$$\text{Area } PBOR = \text{Rectangle } PSOR - \text{Area } PBS;$$

therefore

$$\int u\,dv = uv - \int v\,du.$$

Applications.

61. In general there will be more than one possible method of selecting the factors u and dv. The latter of course includes the factor dx, but it will generally be advisable to include in it any other factors which permit the direct integration of dv. After selecting the factors, it will be found convenient at once to write the product $u \cdot v$, separating the factors by a period; this will serve as a guide in forming the product

$v\,du$, which is to be written under the integral sign. Thus, let the given integral be

$$\int x^2 \log x\, dx.$$

Taking $x^2 dx$ as the value of dv, since we can integrate this expression directly, we have

$$\int x^2 \log x\, dx = \log x \cdot \frac{1}{3} x^3 - \frac{1}{3}\int x^3 \frac{dx}{x}$$

$$= \frac{1}{3} x^3 \log x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{9}(3 \log x - 1).$$

62. The form of the new integral may be such that a second application of the formula is required before a directly integrable form is produced. For example, let the given integral be

$$\int x^2 \cos x\, dx.$$

In this case we take $\cos x\, dx = dv$; so that having $x^2 = u$, the new integral will contain a lower power of x: thus

$$\int x^2 \cos x\, dx = x^2 \cdot \sin x - 2\int x \sin x\, dx.$$

Making a second application of the formula, we have

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\left[x(-\cos x) + \int \cos x\, dx\right]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x.$$

63. The method of integration by parts is sometimes employed with advantage, even when the new integral is no simpler than the given one; for, in the process of successive applications of the formula, the original integral may be reproduced, as in the following example:

$$\int e^{mx} \sin(nx + \alpha)\, dx$$

$$= e^{mx} \cdot \frac{-\cos(nx+\alpha)}{n} + \frac{m}{n}\int e^{mx}\cos(nx+\alpha)\, dx$$

$$= -\frac{e^{mx}\cos(nx+\alpha)}{n} + \frac{m}{n}\, e^{mx}\frac{\sin(nx+\alpha)}{n} - \frac{m^2}{n^2}\int e^{mx}\sin(nx+\alpha)\, dx,$$

in which the integral in the second member is identical with the given integral; hence, transposing and dividing,

$$\int e^{mx}\sin(nx+\alpha)\, dx = \frac{e^{mx}}{m^2+n^2}[m\sin(nx+\alpha) - n\cos(nx+\alpha)].$$

64. In some cases it is necessary to employ some other mode of transformation, in connection with the method of parts. For example, given the integral

$$\int \sec^3\theta\, d\theta;$$

taking $dv = \sec^2\theta\, d\theta$, we have

$$\int \sec^3\theta\, d\theta = \sec\theta\cdot\tan\theta - \int \sec\theta\tan^2\theta\, d\theta. \quad . \quad . \quad (1)$$

If now we apply the method of parts to the new integral, by putting

$$\sec\theta\tan\theta\,d\theta = dv,$$

the original integral will indeed be reproduced in the second member; but it will disappear from the equation, the result being an identity. If, however, in equation (1), we transform the final integral by means of the equation $\tan^2\theta = \sec^2\theta - 1$, we have

$$\int\sec^3\theta\,d\theta = \sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta.$$

Transposing,

$$2\int\sec^3\theta\,d\theta = \frac{\sin\theta}{\cos^2\theta} + \int\frac{d\theta}{\cos\theta}d\theta:$$

hence, by formula (F), Art. 31,

$$\int\sec^3\theta\,d\theta = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\log\tan\left[\frac{\pi}{4} + \frac{\theta}{2}\right].$$

Formulæ of Reduction.

65. It frequently happens that the new integral introduced by applying the method of parts differs from the given integral only in the values of certain constants. If these constants are expressed algebraically, the formula expressing the first transformation is adapted to the successive transformations of the new integrals introduced, and is called *a formula of reduction.*

For example, applying the method of parts to the integral

$$\int x^n e^{ax}\, dx,$$

we have

$$\int x^n e^{ax}\, dx = x^n \cdot \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx, \quad . \quad . \quad . \quad . \quad (1)$$

in which the new integral is of the same form as the given one, the exponent of x being decreased by unity. Equation (1) is therefore a formula of reduction for this function. Supposing n to be a positive integer, we shall finally arrive at the integral $\int e^{ax}\, dx$, whose value is $\frac{e^{ax}}{a}$. Thus, by successive application of equation (1) we have

$$\int x^n e^{ax}\, dx = \frac{e^{ax}}{a}\left[x^n - \frac{n}{a}\, x^{n-1} \cdot \ldots + (-1)^n \frac{n(n-1)\cdot \ldots \cdot 1}{a^n}\right].$$

Reduction of $\int \sin^m \theta\, d\theta$ *and* $\int \cos^m \theta\, d\theta$.

66. To obtain a formula of reduction, it is sometimes necessary to make a further transformation of the equation obtained by the method of parts. Thus, for the integral

$$\int \sin^m \theta\, d\theta,$$

the method of parts gives

$$\int \sin^m \theta\, d\theta = \sin^{m-1}\theta\,(-\cos\theta) + (m-1)\int \sin^{m-2}\theta \cos^2\theta\, d\theta.$$

Substituting in the latter integral $1 - \sin^2\theta$ for $\cos^2\theta$,

$$\int \sin^m\theta\, d\theta = -\sin^{m-1}\theta\cos\theta + (m-1)\int \sin^{m-2}\theta\, d\theta - (m-1)\int \sin^m\theta\, d\theta;$$

transposing and dividing, we have

$$\int \sin^m\theta\, d\theta = -\frac{\sin^{m-1}\theta\cos\theta}{m} + \frac{m-1}{m}\int \sin^{m-2}\theta\, d\theta, \quad . \quad . \quad . \quad (1)$$

a formula of reduction in which the exponent of $\sin\theta$ is diminished two units. By successive application of this formula, we have, for example:

$$\int \sin^6\theta\, d\theta = -\frac{\sin^5\theta\cos\theta}{6} + \frac{5}{6}\int \sin^4\theta\, d\theta$$

$$= -\frac{\sin^5\theta\cos\theta}{6} - \frac{5}{6}\frac{\sin^3\theta\cos\theta}{4} + \frac{5}{6}\frac{3}{4}\int \sin^2\theta\, d\theta$$

$$= -\frac{\sin^5\theta\cos\theta}{6} - \frac{5\sin^3\theta\cos\theta}{6\cdot 4} - \frac{5\cdot 3\sin\theta\cos\theta}{6\cdot 4\cdot 2} + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\theta.$$

67. By a process similar to that employed in deriving equation (1), or simply by putting $\theta = \frac{1}{2}\pi - \theta'$ in that equation, we find

$$\int \cos^m\theta\, d\theta = \frac{\cos^{m-1}\theta\sin\theta}{m} + \frac{m-1}{m}\int \cos^{m-2}\theta\, d\theta, \quad . \quad . \quad (2)$$

a formula of reduction, when m is positive.

68. It should be noticed that, when m is negative, equation (1) Art. 66 is not a formula of reduction, because the exponent in the new integral is in that case numerically greater than the exponent in the given integral. But, if we now regard the integral in the second member as the given one, the equation is readily converted into a formula of reduction. Thus, putting $-n$ for the negative exponent $m-2$, whence

$$m = -n + 2,$$

transposing and dividing, equation (1) becomes

$$\int \frac{d\theta}{\sin^n \theta} = -\frac{\cos\theta}{(n-1)\sin^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta}. \quad . \quad . \quad . \quad (3)$$

Again, putting $\theta = \frac{1}{2}\pi - \theta'$ in this equation, we obtain

$$\int \frac{d\theta}{\cos^n \theta} = \frac{\sin\theta}{(n-1)\cos^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\cos^{n-2}\theta} \quad . \quad . \quad . \quad . \quad (4)$$

Reduction of $\int \sin^m\theta \cos^n\theta\, d\theta$.

69. If we put $dv = \sin^m\theta\cos\theta\, d\theta$, we have

$$\int \sin^m\theta\cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}\theta\cos^{n-2}\theta\, d\theta; \quad . \quad . \quad . \quad (1)$$

but, if in the same integral we put $dv = \cos^n\theta\sin\theta\, d\theta$, we have

$$\int \sin^m\theta\cos^n\theta\, d\theta = -\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}\theta\cos^{n+2}\theta\, d\theta. \quad . \quad . \quad . \quad (2)$$

When m and n are both positive, equation (1) is not a formula of reduction, since in the new integral the exponent of $\sin\theta$ is increased, while that of $\cos\theta$ is diminished. We therefore substitute in this integral

$$\sin^{m+2}\theta = \sin^m\theta(1 - \cos^2\theta),$$

so that the last term of the equation becomes

$$\frac{n-1}{m+1}\int \sin^m\theta\cos^{n-2}\theta\,d\theta - \frac{n-1}{m+1}\int \sin^m\theta\cos^n\theta\,d\theta.$$

Hence, by this transformation, the original integral is reproduced, and equation (1) becomes

$$\left[1 + \frac{n-1}{m+1}\right]\int \sin^m\theta\cos^n\theta\,d\theta = \frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^m\theta\cos^{n-2}\theta\,d\theta.$$

Dividing by $1 + \frac{n-1}{m+1} = \frac{m+n}{m+1}$, we have

$$\int \sin^m\theta\cos^n\theta\,d\theta = \frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m\theta\cos^{n-2}\theta\,d\theta, \quad . \quad . \quad . \quad (3)$$

a formula of reduction by which the exponent of $\cos\theta$ is diminished two units.

By a similar process, from equation (2), or simply by putting $\theta = \frac{1}{2}\pi - \theta'$ in equation (3), and interchanging m and n, we obtain

$$\int \sin^m \theta \cos^n \theta \, d\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n \theta \, d\theta, \quad . \quad . \quad . \quad (4)$$

a formula by which the exponent of $\sin\theta$ is diminished two units.

70. When n is positive and m negative, equation (1) of the preceding article is itself a formula of reduction, for both exponents are in that case numerically diminished. Putting $-m$ in place of m, the equation becomes

$$\int \frac{\cos^n \theta}{\sin^m \theta} d\theta = -\frac{\cos^{n-1}\theta}{(m-1)\sin^{m-1}\theta} - \frac{n-1}{m-1}\int \frac{\cos^{n-2}\theta}{\sin^{m-2}\theta} d\theta. \quad . \quad . \quad (5)$$

Similarly, when m is positive and n negative, equation (2) gives

$$\int \frac{\sin^m \theta}{\cos^n \theta} d\theta = \frac{\sin^{m-1}\theta}{(n-1)\cos^{n-1}\theta} - \frac{m-1}{n-1}\int \frac{\sin^{m-2}\theta}{\cos^{n-2}\theta} d\theta. \quad . \quad . \quad . \quad (6)$$

71. When m and n are both negative, putting $-m$ and $-n$ in place of m and n, equation (3) Art. 69 becomes

$$\int \frac{d\theta}{\sin^m \theta \cos^n \theta} = -\frac{1}{(m+n)\sin^{m-1}\theta \cos^{n+1}\theta} + \frac{n+1}{m+n}\int \frac{d\theta}{\sin^m \theta \cos^{n+2}\theta},$$

in which the exponent of $\cos\theta$ is numerically increased. We

may therefore regard the integral in the second member as the integral to be reduced. Thus, putting n in place of $n + 2$, we derive

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{1}{(n-1)\sin^{m-1}\theta\cos^{n-1}\theta} + \frac{m+n-2}{n-1}\int \frac{d\theta}{\sin^m\theta\cos^{n-2}\theta} \quad \ldots\ldots (7)$$

Putting $\theta = \frac{1}{2}\pi - \theta'$, and interchanging m and n, we have

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = -\frac{1}{(m-1)\sin^{m-1}\theta\cos^{n-1}\theta} + \frac{m+n-2}{m-1}\int \frac{d\theta}{\sin^{m-2}\theta\cos^n\theta} \quad \ldots\ldots (8)$$

72. The application of the formulæ derived in the preceding articles to definite integrals will be given in the next section. When the value of the indefinite integral is required, it should first be ascertained whether the given integral belongs to one of the directly integrable cases mentioned in Arts. 27 and 28. If it does not, the formulæ of reduction must be used, and if m and n are integers, we shall finally arrive at a directly integrable form.

As an illustration, let us take the integral

$$\int \sin^2\theta\cos^4\theta\, d\theta.$$

Employing formula (4) Art. 69, by which the exponent of $\sin\theta$ is diminished, we have

$$\int \sin^2\theta\cos^4\theta\, d\theta = -\frac{\sin\theta\cos^5\theta}{6} + \frac{1}{6}\int \cos^4\theta\, d\theta.$$

The last integral can be reduced by means of formula (2) Art. 67, which, when $m = 4$, gives

$$\int \cos^4\theta\, d\theta = \frac{\cos^3\theta \sin\theta}{4} + \frac{3}{4}\int \cos^2\theta\, d\theta;$$

therefore

$$\int \sin^2\theta \cos^4\theta\, d\theta = -\frac{\sin\theta\cos^5\theta}{6} + \frac{\cos^3\theta\sin\theta}{24} + \frac{\sin\theta\cos\theta}{16} + \frac{\theta}{16}.$$

73. Again, let the given integral be

$$\int \frac{\cos^6\theta\, d\theta}{\sin^3\theta}.$$

By equation (5), Art. 70, we have

$$\int \frac{\cos^6\theta\, d\theta}{\sin^3\theta} = -\frac{\cos^5\theta}{2\sin^2\theta} - \frac{5}{2}\int \frac{\cos^4\theta\, d\theta}{\sin\theta}.$$

We cannot apply the same formula to the new integral, since the denominator $m - 1$ vanishes; but putting $n = 4$ and $m = -1$, in equation (3) Art. 69, we have

$$\int \frac{\cos^4\theta\, d\theta}{\sin\theta} = \frac{\cos^3\theta}{3} + \int \frac{\cos^2\theta\, d\theta}{\sin\theta}$$

$$= \frac{\cos^3\theta}{3} + \int \frac{d\theta}{\sin\theta} - \int \sin\theta\, d\theta$$

$$= \frac{\cos^3\theta}{3} + \log\tan\frac{1}{2}\theta + \cos\theta.$$

Hence

$$\int \frac{\cos^6\theta\, d\theta}{\sin^3\theta} = -\frac{\cos^5\theta}{2\sin^2\theta} - \frac{5\cos^3\theta}{6} - \frac{5}{2}\log\tan\frac{1}{2}\theta - \frac{5}{2}\cos\theta.$$

Reduction of $\int \cos^m\theta \cos n\theta \, d\theta.$

74. Integrating by parts, we have

$$\int \cos^m\theta \cos n\theta d\theta = \cos^m\theta \cdot \frac{\sin n\theta}{n} + \frac{m}{n}\int \sin n\theta \cos^{m-1}\theta \sin\theta d\theta.$$

To obtain a formula of reduction we employ the identity

$$\sin n\theta \sin\theta = \cos(n-1)\theta - \cos n\theta \cos\theta,$$

so that the last term of the equation becomes

$$\frac{m}{n}\int \cos^{m-1}\theta \cos(n-1)\theta d\theta - \frac{m}{n}\int \cos^m\theta \cos n\theta d\theta.$$

The original integral is thus reproduced, and after transposition and division the equation becomes

$$\int \cos^m\theta \cos n\theta d\theta = \frac{\cos^m\theta \sin n\theta}{m+n} + \frac{m}{m+n}\int \cos^{m-1}\theta \cos(n-1)\theta d\theta, \quad (1)$$

a formula of reduction in which m and n are each diminished one unit.

If m is a positive integer, we shall by repeated applications of this formula arrive at a directly integrable form, even when n is not an integer; and it may be remarked that, since $\cos(-n\theta) = \cos n\theta$, the sign of n may at any time be changed if convenient. As an example, we have

$$\int \cos^2\theta \cos\tfrac{1}{3}\theta d\theta = \frac{\cos^2\theta \sin\frac{1}{3}\theta}{2\frac{1}{3}} + \frac{2}{2\frac{1}{3}}\int \cos\theta \cos\tfrac{2}{3}\theta d\theta$$

$$= \tfrac{3}{7}\cos^2\theta \sin\tfrac{1}{3}\theta + \frac{6}{7}\left[\frac{\cos\theta \sin\frac{2}{3}\theta}{1\frac{2}{3}} + \frac{3}{5}\int \cos\tfrac{1}{3}\theta d\theta\right]$$

$$= \tfrac{3}{7}\cos^2\theta \sin\tfrac{1}{3}\theta + \tfrac{18}{35}\cos\theta \sin\tfrac{2}{3}\theta + \tfrac{54}{35}\sin\tfrac{1}{3}\theta.$$

75. If m be negative, putting $m = -m', n = -n'$, equation (1) becomes

$$\int \frac{\cos n'\theta}{\cos^{m'}\theta} d\theta = \frac{\sin n'\theta}{(m'+n')\cos^{m'}\theta} + \frac{m'}{m'+n'} \int \frac{\cos (n'+1)\theta}{\cos^{m'+1}\theta} d\theta,$$

in which we may regard the integral in the second member as the one to be reduced. Hence, putting n for $n'+1$ and m for $m'+1$, we derive

$$\int \frac{\cos n\theta}{\cos^{m}\theta} d\theta = -\frac{\sin (n-1)\theta}{(m-1)\cos^{m-1}\theta} + \frac{m+n-2}{m-1} \int \frac{\cos (n-1)\theta}{\cos^{m-1}\theta} d\theta. \quad (2)$$

If m is an integer, we can by this formula make the integration depend upon a final integral of the form $\int \frac{\cos p\theta}{\cos \theta} d\theta$, which is readily evaluated when n (and therefore p) is an integer, and also sometimes when n is a fraction. For example,

$$\begin{aligned}
\int \frac{\cos \frac{3}{2}\theta}{\cos^3 \theta} d\theta &= -\frac{\sin \frac{1}{2}\theta}{2\cos^2 \theta} + \frac{2\frac{1}{2}}{2} \int \frac{\cos \frac{1}{2}\theta}{\cos^2 \theta} d\theta \\
&= -\frac{\sin \frac{1}{2}\theta}{2\cos^2 \theta} + \frac{5}{4}\left[-\frac{\sin (-\frac{1}{2}\theta)}{\cos \theta} + \frac{1}{2} \int \frac{\cos \frac{1}{2}\theta}{\cos \theta} d\theta \right] \\
&= -\frac{\sin \frac{1}{2}\theta}{2\cos^2 \theta} + \frac{5 \sin \frac{1}{2}\theta}{4\cos \theta} + \frac{5}{4} \int \frac{\cos \phi d\phi}{\cos 2\phi} \\
&= -\frac{\sin \frac{1}{2}\theta}{2\cos^2 \theta} + \frac{5 \sin \frac{1}{2}\theta}{4\cos \theta} + \frac{5}{8\sqrt{2}} \log \frac{1+\sqrt{2}\sin \frac{1}{2}\theta}{1-\sqrt{2}\sin \frac{1}{2}\theta}.
\end{aligned}$$

Formulæ of reduction may be deduced in like manner for the integration of $\cos^m \theta \sin n\theta d\theta$, and similar expressions.

Reduction of Algebraic Forms.

76. The method of reduction may be applied with advantage to some algebraic forms. Take for example the form

$$\int \frac{x^m dx}{(a+bx^2)^n},$$

in which m and n are supposed positive. Integrating by parts, taking $u = x^{m-1}$ and

$$dv = \frac{x\,dx}{(a + bx^2)^n}, \quad \text{whence} \quad v = -\frac{1}{2b(n-1)(a+bx^2)^{n-1}},$$

we have

$$\int \frac{x^m dx}{(a+bx^2)^n} = -\frac{x^{m-1}}{2b(n-1)(a+bx^2)^{n-1}} + \frac{m-1}{2b(n-1)} \int \frac{x^{m-2}dx}{(a+bx^2)^{n-1}}$$

a formula of reduction in which m is reduced by two units and n by one unit.

If m and n are integers, the expression to be integrated is a rational fraction and includes (after a simple transformation) all cases of multiple imaginary roots. Repeated applications of the formula will reduce n to unity or else reduce m to zero or to unity, the final integral vanishing in the last case. For example, the integral occurring in Art. 23 is thus reduced to

$$\int \frac{a^3 + 3a^2x + ax^2 + x^3}{(a^2+x^2)^2} dx = -\frac{x^2}{2(a^2+x^2)} + \tfrac{1}{2}\log(a^2+x^2)$$
$$- \frac{ax}{2(a^2+x^2)} + \tfrac{1}{2}\tan^{-1}\frac{x}{a} - \frac{3a^2}{2(a^2+x^2)} + \int \frac{a^3 dx}{(a^2+x^2)^2},$$

in which the final integral only requires further transformation; the result is

$$\int \frac{a^3 + 3a^2x + ax^2 + x^3}{(a^2+x^2)^2} dx = -\frac{3a^2+x^2}{2(a^2+x^2)} + \tfrac{1}{2}\log(a^2+x^2) + \tan^{-1}\frac{x}{a}.$$

The final integral is also of an integrable form when n is half of an odd integer.

General Method of Deriving a Formula of Reduction.

77. The method of integration by parts may be said to consist in finding a function, uv, of which the differential con-

tains, as one part, the differential to be integrated; so as to make the integration depend upon that of the other part. If this last is of the same general form as the given differential, we have a formula of reduction.

The more general method consists in finding a function, P, whose differential can be expressed as the sum of certain multiples of two or more differential expressions of a given general form. We then have an equation connecting P with two or more integrals of the given general form. Among these we select that of the highest degree as the one to be reduced, and so prepare a formula of reduction for the given integral form.

For example, to find a formula of reduction for the form

$$\int \frac{dx}{(a+2bx+cx^2)^n}, \quad \text{or} \quad \int \frac{dx}{X^n},$$

where X is a quadratic expression which for convenience is written in the form

$$X = a + 2bx + cx^2,$$

whence

$$dX = 2(b+cx)dx.$$

If we take X^{-m} for P, we shall find it impossible to reduce dP to the required form. The same is true if we take xX^{-m}; but by a combination of these forms we are able to accomplish our object. Thus, taking

$$P = \frac{p+qx}{X^m}$$

(in which p and q are constants to be determined), we have

$$dP = \frac{q\,dx}{X^m} - \frac{m(p+qx)dX}{X^{m+1}} = \frac{q\,dx}{X^m} - 2m\frac{(p+qx)(b+cx)dx}{X^{m+1}}.$$

The first term is of the proposed form, and the second will

be so if p and q be so taken as to make $(p + qx)(b + cx)$ a multiple of X plus a constant. This will be the case if we put $q = c$ and $p = b$, which gives

$$(p + qx)(b + cx) = (b + cx)^2 = cX - ac + b^2.$$

Hence we have $$P = \frac{b + cx}{X^m},$$

and
$$dP = \frac{cdx}{X^m} - 2m\frac{cX - ac + b^2}{X^{m+1}}dx$$

$$= c(1 - 2m)\frac{dx}{X^m} + 2m(ac - b^2)\frac{dx}{X^{m+1}},$$

which is in the required form. Since the last expression is of the higher degree in X, we put $m + 1 = n$, and obtain

$$\int\frac{dx}{X^n} = \frac{b + cx}{2(n - 1)(ac - b^2)X^{n-1}} + \frac{(2n - 3)c}{2(n - 1)(ac - b^2)}\int\frac{dx}{X^{n-1}},$$

the formula of reduction required.

When n is an integer, the final integral, in successive applications of this formula, is reducible to one of the forms (A), p. 18, or (k'), p. 9; and when n is half of an odd integer, it disappears by virtue of the factor $2n - 3$, the last integration being equivalent to the use of formula (J), p. 63.

Development of an Integral in Series.

78. It is often desirable to express an integral in the form of a series involving powers of the independent variable x, especially in the case of those integrals which cannot be ex-

pressed by means of the elementary functions, that is to say, expressions which we now regard as not integrable.

The most obvious way to do this is to develop into a series the expression under the integral sign, and then to integrate term by term. Given for example $\int \frac{e^x}{x} dx$. Using the expansion of e^x, we have

$$\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{r-1}}{r!} + \ldots$$

Hence

$$\int \frac{e^x}{x} dx = C + \log x + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \ldots + \frac{x^r}{r \cdot r!} + \ldots$$

The series integrated is, in this case, convergent for all values of x, from which it readily follows that the result of integration is also a convergent series for all values of x.

If we put

$$F(x) = \int f(x) dx,$$

and $f(x)$ can be developed by Maclaurin's Theorem, we have

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots,$$

and integrating,

$$F(x) = \int f(x) dx = C + f(0)x + f'(0)\frac{x^2}{2!} + f''(0)\frac{x^3}{3!} + \ldots, \quad (1)$$

in which $C = F(0)$. The result is the same as the development of $F(x)$ by Maclaurin's Theorem, since $f(x) = F'(x)$, $f'(x) = F''(x) \ldots$

Bernoulli's Series.

79. By successive applications of integration by parts, we have

$$\int_0^x f(z)dz = xf(x) - \int_0^x zf'(z)dz$$

$$= xf(x) - \frac{x^2}{2}f'(x) + \int_0^x \frac{z^2}{2}f''(z)dz;$$

. . . . . .

and finally, provided neither $f(z)$ nor any of its derivatives to the nth inclusive becomes infinite for $z = 0$, $z = x$ or any intermediate value of z,

$$\int_0^x f(z)dz = xf(x) - \frac{x^2}{2!}f'(x) + \frac{x^3}{3!}f''(x) - \ldots$$

$$- (-1)^n \frac{x^n}{n!}f^{n-1}(x) + (-1)^n \int_0^x \frac{z^n}{n!}f^n(z)dz \ldots, \quad (2)$$

which is known as Bernoulli's Series. It is not, in the ordinary sense, a development in powers of x, because the coefficients themselves contain x; they are in fact the values of $f(x)$ and its derivatives at the upper limit, instead of their values at the lower limit, as in equation (1) of the preceding article. The result may in fact be derived from that equation when written in the form

$$\int_0^x f(z)dz = f(0)x + f'(0)\frac{x^2}{2!} + f''(0)\frac{x^3}{3!} + \ldots \quad . \quad . \quad (3)$$

For, putting $z = x - y$, and denoting $f(x - y)$ by $\phi(y)$, we find

$$\int_0^x f(z)dz = -\int_x^0 f(x-y)dy = \int_0^x \phi(y)dy.$$

Developing the last integral by equation (3), we have

$$\int_0^x f(z)dz = \int_0^x \phi(y)dy = \phi(0)x + \phi'(0)\frac{x^2}{2!} + \phi''(0)\frac{x^3}{3!} + \ldots,$$

or, since $\phi'(y) = -f'(x-y)$, $\phi''(y) = f''(x-y)\ldots$, so that $\phi(0) = f(x)$, $\phi'(0) = -f'(x)$, $\phi''(0) = f''(x)\ldots$,

$$\int_0^x f(z)dz = f(x)x - f'(x)\frac{x^2}{2!} + f''(x)\frac{x^3}{3!} - \ldots,$$

which is Bernoulli's Series.

Taylor's Theorem.

80. Applying Bernoulli's Series, equation (2) of the preceding article, to the function

$$f(z) = F'(x_0 + h - z);$$

whence

$$f'(z) = -F''(x_0 + h - z), \quad f''(z) = F'''(x_0 + h - z), \ldots;$$

and using h for the upper limit, we have

$$\int_0^h F'(x_0+h-z)dz = hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \frac{h^3}{3!}F'''(x_0) + \ldots + \frac{h^n}{n!}F^n(x_0) + \int_0^h \frac{z^n}{n!}F^{n+1}(x_0+h-z)dz.$$

But, denoting by F the function of which F' is the derivative, the value of the first member is $F(x_0 + h) - (Fx_0)$; therefore

$$F(x_0+h) = F(x_0) + hF'(x_0) \ldots + \frac{h^n}{n!}F^n(x_0) + \frac{1}{n!}\int_0^h F^{n+1}(x_0+h-z)z^n dz,$$

which is Taylor's Theorem with the remainder expressed in the form of a definite integral.

Examples VI.

1. $\int_0^1 \sin^{-1} x\, dx$, $\quad x \sin^{-1} x + \sqrt{(1 - x^2)} \Big]_0^1 = \frac{\pi}{2} - 1.$

2. $\int \sec^{-1} x\, dx$, $\quad x \sec^{-1} x - \log [x + \sqrt{(x^2 - 1)}].$

3. $\int_0^1 \tan^{-1} x\, dx$, $\quad \frac{\pi}{4} - \frac{\log 2}{2}.$

4. $\int x^n \log x\, dx$, $\quad \frac{x^{n+1}}{n+1}\left[\log x - \frac{1}{n+1}\right].$

5. $\int_0^{\frac{\pi}{2}} \theta \sin \theta\, d\theta$, $\quad 1.$

6. $\int_0^{\frac{\pi}{2}} \theta \cos m\theta\, d\theta$, $\quad \frac{\pi}{2m} \sin \frac{m\pi}{2} - \frac{2}{m^2} \sin^2 \frac{m\pi}{4}.$

7. $\int x \tan^{-1} x\, dx$, $\quad \frac{1 + x^2}{2} \tan^{-1} x - \frac{x}{2}.$

8. $\int_0^{\infty} x^2 \epsilon^x\, dx$, $\quad x^2 e^x - 2xe^x + 2e^x - 2.$

9. $\int x \sec^{-1} x\, dx$, $\quad \tfrac{1}{2}[x^2 \sec^{-1} x - \sqrt{(x^2 - 1)}].$

10. $\int_0^{\frac{\pi}{2}} \theta \sin\left[\frac{\pi}{4} + \theta\right] d\theta$, $\quad -\theta \cos\left(\frac{\pi}{4} + \theta\right) + \sin\left(\frac{\pi}{4} + \theta\right) \Big]_0^{\frac{\pi}{2}} = \frac{\pi\sqrt{2}}{4}.$

11. $\int x \sec^2 x\, dx$, $\qquad x \tan x + \log \cos x.$

12. $\int x \tan^2 x\, dx \left[= \int x\, (\sec^2 x - 1)\, dx \right]$, $x \tan x + \log \cos x - \frac{1}{2} x^2.$

13. $\int x^2 \sin x\, dx$, $\qquad 2x \sin x + 2 \cos x - x^2 \cos x.$

14. $\int_0^1 x \sin^{-1} x\, dx$, $\qquad \frac{1}{2} x^2 \sin^{-1} x \Big]_0^1 - \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \theta\, d\theta = \frac{\pi}{8}.$

15. $\int x^2 \tan^{-1} x\, dx$, $\qquad \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{\log (1 + x^2)}{6}.$

16. $\int_0^1 x^2 \sin^{-1} x\, dx$, $\quad \frac{1}{3} x^3 \sin^{-1} x + \frac{2 + x^2}{9} \sqrt{(1 - x^2)} \Big]_0^1 = \frac{\pi}{6} - \frac{2}{9}$

17. $\int_0^\infty e^{-x} \cos x\, dx$, $\qquad \frac{e^{-x} (\sin x - \cos x)}{2} \Big]_0^\infty = \frac{1}{2}.$

18. $\int e^{x \tan \beta} \cos x\, dx$, $\qquad \cos \beta\, e^{x \tan \beta} \sin (\beta + x).$

19. $\int e^{-x} \sin^2 x\, dx \left[= \frac{1}{2} \int e^{-x} (1 - \cos 2x)\, dx \right]$,

$$\frac{e^{-x}}{10} (\cos 2x - 2 \sin 2x - 5).$$

20. $\int_0^{\frac{\pi}{4}} e^\theta \sin \theta\, d\theta$, $\qquad \frac{e^\theta}{2} (\sin \theta - \cos \theta) \Big]_0^{\frac{\pi}{4}} = \frac{1}{2}.$

21. $\int e^x \sin x \cos x \, dx$, $\qquad \frac{e^x}{10}(\sin 2x - 2\cos 2x)$.

22. $\int_0 \sin^4 m\theta \, d\theta$, $\qquad \frac{-\sin^3 m\theta \cos m\theta}{4m} + \frac{3\theta}{8} - \frac{3 \sin m\theta \cos m\theta}{8m}$.

23. Derive a formula of reduction for $\int (\log x)^n x^m \, dx$, and deduce from it the value of $\int (\log x)^3 x^2 \, dx$.

$$\int (\log x)^n x^m \, dx = (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int (\log x)^{n-1} x^m \, dx.$$

$$\int (\log x)^3 x^2 \, dx = (\log x)^3 \frac{x^3}{3} - (\log x)^2 \frac{x^3}{3} + \frac{2x^3 \log x}{9} - \frac{2x^3}{27}.$$

24. $\int x \cos^2 x \, dx$, $\qquad \frac{1}{2} x \sin x \cos x - \frac{1}{4} \sin^2 x + \frac{1}{4} x^2$.

25. $\int_1 x^2 \sec^{-1} x \, dx$, $\qquad \frac{x^3 \sec^{-1} x}{3} - \frac{x\sqrt{(x^2-1)}}{6} - \frac{\log [x + \sqrt{(x^2-1)}]}{6}$.

26. Derive a formula of reduction for $\int x^n \sin(x + \alpha) \, dx$, and deduce from it the value of $\int x^5 \cos x \, dx$.

$$\int x^n \sin(x + \alpha) \, dx = -x^n \sin\left[x + \alpha + \frac{\pi}{2}\right] + n \int x^{n-1} \sin\left[x + \alpha + \frac{\pi}{2}\right] dx.$$

$$\int x^5 \cos x \, dx = (x^5 - 20x^3 + 120x) \sin x + (5x^4 - 60x^2 + 120) \cos x.$$

27. $\int \cos^2\theta \sin^4\theta \, d\theta$, $\quad \frac{\sin^5\theta\cos\theta}{6} - \frac{\sin^3\theta\cos\theta}{24} + \frac{1}{16}[\theta - \sin\theta\cos\theta]$.

28. $\int_0^{\frac{\pi}{4}} \cos^4\theta \sin^4\theta \, d\theta$, $\quad \frac{1}{32}\int_0^{\frac{\pi}{2}} \sin^4\theta' \, d\theta' = \frac{3\pi}{512}$.

29. $\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta$, $\quad \frac{\sin\theta\cos^3\theta}{4} + \frac{3\sin\theta\cos\theta}{8} + \frac{3\theta}{8}\Big]_0^{\frac{\pi}{4}} = \frac{8+3\pi}{32}$.

30. $\int_0^{\frac{\pi}{3}} \cos^6\theta \, d\theta$,

$$\frac{\sin\theta\cos\theta(8\cos^4\theta + 10\cos^2\theta + 15) + 15\theta}{48}\Big]_0^{\frac{\pi}{3}} = \frac{9\sqrt{3}+10\pi}{96}.$$

31. $\int \frac{\cos^4\theta}{\sin^3\theta} d\theta$, $\quad -\frac{\cos^3\theta}{2\sin^2\theta} - \frac{3\cos\theta}{2} - \frac{3\log\tan\frac{1}{2}\theta}{2}$.

32. $\int \frac{\sin^2\theta}{\cos^5\theta} d\theta$, $\quad \frac{\sin\theta}{4\cos^4\theta} - \frac{\sin\theta}{8\cos^2\theta} - \frac{1}{8}\log\tan\left[\frac{\pi}{4} + \frac{\theta}{2}\right]$.

33. $\int \frac{\sin^5\theta}{\cos^4\theta} d\theta$, $\quad \frac{\sin^5\theta}{3\cos^3\theta} - \frac{5\sin^3\theta}{3\cos\theta} + \frac{5}{2}[\theta - \sin\theta\cos\theta]$.

34. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^6\theta}{\sin^2\theta} d\theta$, $\quad -\frac{\cos^5\theta}{\sin\theta}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - 5\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta \, d\theta = \frac{48-15\pi}{32}$.

35. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+\cos\theta)^2}$, $\quad \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta'}{\cos^4\theta'} = \frac{2}{3}$.

36. $\int \frac{d\theta}{\sin\theta\cos^4\theta}$, $\quad \frac{1}{3\cos^3\theta} + \frac{1}{\cos\theta} + \log\tan\frac{\theta}{2}$.

37. $\int \frac{d\theta}{\sin\theta \sin^2 2\theta}$, $\quad \frac{1}{4\sin^2\theta\cos\theta} - \frac{3}{8}\frac{\cos\theta}{\sin^2\theta} + \frac{3}{8}\log\tan\frac{\theta}{2}$.

38. Prove that when n is odd

$$\int \frac{d\theta}{\sin\theta\cos^n\theta} = \frac{\sec^{n-1}\theta}{n-1} + \frac{\sec^{n-3}\theta}{n-3} + \cdots\cdots + \log\tan\theta;$$

and when n is even

$$\int \frac{d\theta}{\sin\theta\cos^n\theta} = \frac{\sec^{n-1}\theta}{n-1} + \frac{\sec^{n-3}\theta}{n-3} + \cdots\cdots + \log\tan\frac{\theta}{2}.$$

39. $\int \frac{d\theta}{\sin^3\theta\cos^4\theta}$, $\quad -\frac{1}{2\sin^2\theta\cos^3\theta} + \frac{5}{2}\left[\frac{\sec^3\theta}{3} + \sec\theta + \log\tan\frac{\theta}{2}\right]$.

40. $\int \frac{x^2\,dx}{\sqrt{(x^2-1)}}$, $\quad$ *Put* $x = \sec\theta$.

$$\tfrac{1}{2}x\sqrt{(x^2-1)} + \tfrac{1}{2}\log[x + \sqrt{(x^2-1)}].$$

41. $\int (a^2 - x^2)^{\frac{3}{2}}\,dx$, $\quad \frac{(5a^2 - 2x^2)\,x\sqrt{(a^2-x^2)}}{8} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a}$.

42. $\int_0^a \frac{dx}{(a^2+x^2)^3}$, $\quad \frac{1}{a^5}\int_0^{\frac{\pi}{4}} \cos^4\theta\,d\theta = \frac{3\pi + 8}{32a^5}$.

43. $\int (a^2 + x^2)\sqrt{(a^2 - x^2)}\,dx$, $\quad \frac{5a^4}{8}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{(a^2-x^2)}[3a^2+2x^2]}{8}$.

44. $\int \frac{x^2\,dx}{(x^2-a^2)^{\frac{3}{2}}}$, $\quad -\frac{x}{\sqrt{(x^2-a^2)}} + \log[x + \sqrt{(x^2-a^2)}]$.

45. $\int \frac{x^2\,dx}{(x^2+1)^3}$, $\quad \frac{x(x^2-1)}{8(x^2+1)^2} + \frac{\tan^{-1}x}{8}$

46. $\int \frac{\cos^3\theta - \sin^3\theta}{(\sin\theta + \cos\theta)^2}\, d\theta$ $\left[= \int (1 + \sin\theta\cos\theta)\frac{\cos\theta - \sin\theta}{(\sin\theta + \cos\theta)^2}\, d\theta\right]$,

$\frac{\sin\theta\cos\theta}{\sin\theta + \cos\theta}$.

47. Derive a formula for the reduction of $\int x \sec^n x\, dx$; and referring to Ex. 11, thence show that this is an integrable form when n is an even integer. Give the result when $n = 4$.

$$\int x \sec^n x\, dx = \frac{x \sec^{n-2} x \tan x}{n-1} - \frac{\sec^{n-2} x}{(n-1)(n-2)}$$

$$+ \frac{n-2}{n-1}\int x \sec^{n-2} x\, dx.$$

$$\int x \sec^4 x\, dx = \frac{x \sec^2 x \tan x}{3} - \frac{\sec^2 x}{6} + \frac{2}{3}[x \tan x + \log\cos x].$$

48. Derive a formula of reduction for $\int x \cos^n x\, dx$, and deduce from it the value of $\int x \cos^3 x\, dx$.

$$\int x \cos^n x\, dx = \frac{x \cos^{n-1} x \sin x}{n} + \frac{\cos^n x}{n^2} + \frac{n-1}{n}\int x \cos^{n-2} x\, dx.$$

$$\int x \cos^3 x\, dx = \frac{x \sin x}{3}(\cos^2 x + 2) + \frac{\cos x}{9}(\cos^2 x + 6).$$

49. $\int \cos^2\theta \cos 4\theta\, d\theta$, $\frac{\cos^2\theta \sin 4\theta}{6} + \frac{\cos\theta \sin 3\theta}{12} - \frac{\sin 2\theta}{24}$.

50. $\int \cos^4\theta \cos 3\theta\, d\theta$,

$$\frac{\cos^4\theta \sin 3\theta}{7} + \frac{4\cos^3\theta \sin 2\theta}{35} + \frac{12\sin\theta}{35} - \frac{4\sin^3\theta}{35}.$$

51. $\int \cos^3\theta \cos\tfrac{1}{2}\theta\, d\theta$, $\frac{2\sin\frac{1}{2}\theta}{7}\left[\cos^3\theta + \frac{6\cos^2\theta}{5} + \frac{8}{5}\cos\theta + \frac{16}{5}\right]$.

52. $\int \frac{\cos 3\theta\, d\theta}{\cos^2\theta}$, $4\theta - 3\tan\theta$.

53. $\int \frac{\cos\frac{1}{3}\theta\, d\theta}{\cos^2\theta}$, $\frac{\sin\frac{2}{3}\theta}{\cos\theta} + \frac{1}{6}\log\frac{(1+\sin\frac{1}{3}\theta)(1+2\sin\frac{1}{3}\theta)}{(1-\sin\frac{1}{3}\theta)(1-2\sin\frac{1}{3}\theta)}$.

54. Derive a formula of reduction for $\int \cos^m\theta \sin n\theta\, d\theta$.

$$\int \cos^m\theta \sin n\theta\, d\theta = -\frac{\cos^m\theta\cos n\theta}{m+n} + \frac{m}{m+n}\int \cos^{m-1}\theta \sin(n-1)\theta\, d\theta.$$

55. $\int \sqrt{\cos\theta}\sin 3\theta\, d\theta$, $-\frac{2}{7}\sqrt{\cos\theta}\,.\cos 3\theta - \frac{8}{21}(\cos\theta)^{\frac{3}{2}}$.

56. Derive a formula of reduction for $\int \frac{\sin n\theta\, d\theta}{\cos^m\theta}$,

$$\int \frac{\sin n\theta\, d\theta}{\cos^m\theta} = \frac{\cos(n-1)\theta}{(m-1)\cos^{m-1}\theta} + \frac{m+n-2}{m-1}\int \frac{\sin(n-1)\theta\, d\theta}{\cos^{m-1}\theta}.$$

57. Derive the formula of reduction for $\int \frac{dx}{(a+bx^2)^n}$ by the method of Art. 77, and deduce the value of $\int \frac{dx}{(1-x^2)^4}$.

$$\int \frac{dx}{(a+bx^2)^n} = \frac{x}{2a(n-1)(a+bx^2)^{n-1}} + \frac{2n-3}{2a(n-1)}\int \frac{dx}{(a+bx^2)^{n-1}}.$$

$$\int \frac{dx}{(1-x^2)^4} = \frac{x}{6(1-x^2)^3} + \frac{5x}{24(1-x^2)^2} + \frac{5x}{16(1-x^2)} + \frac{5}{32}\log\frac{1+x}{1-x}.$$

58. $\int \frac{x^6 dx}{(1-x^2)^6}$, $\frac{x^5}{10(1-x^2)^5} - \frac{x^3}{16(1-x^2)^4} + \frac{x}{32(1-x^2)^3}$

$$- \frac{x}{128(1-x^2)^2} - \frac{3x}{256(1-x^2)} - \frac{3}{512}\log\frac{1+x}{1-x}.$$

59. $\int \frac{x^2 dx}{(1+x^2)^4}$, $- \frac{x}{6(1+x^2)^3} + \frac{x}{24(1+x^2)^2}$

$$+ \frac{x}{16(1+x^2)} + \frac{\tan^{-1}x}{16}.$$

60. Evaluate $\int \frac{x^5 dx}{(1+x^2)^6}$, ($\alpha$) by the formulæ of reduction; (β) by substituting $y = 1 + x^2$. Show that the results are identical.

(α) $$\int \frac{x^5 dx}{(1+x^2)^6} = - \frac{x^4}{10(1+x^2)^5} - \frac{x^2}{20(1+x^2)^4} - \frac{1}{60(1+x^2)^3}.$$

(β) $$\int \frac{x^5 dx}{(1+x^2)^6} = - \frac{1}{10(1+x^2)^5} + \frac{1}{4(1+x^2)^4} - \frac{1}{6(1+x^2)^3}.$$

61. $\int \frac{x^4 dx}{(x^2-a^2)^{\frac{5}{2}}}$, $\frac{x(3a^2-4x^2)}{3(x^2-a^2)^{\frac{3}{2}}} + \log[x + \sqrt{(x^2-a^2)}]$.

62. $\int \frac{dx}{(x^2-x+1)^3}$,

$$\frac{2x-1}{6(x^2-x+1)^2} + \frac{2x-1}{3(x^2-x+1)} + \frac{4}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

63. $\int \frac{dx}{(x^2-2x\cos\alpha+1)^{\frac{5}{2}}}$,

$$\frac{x-\cos\alpha}{3\sin^2\alpha(x^2-2x\cos\alpha+1)^{\frac{3}{2}}} + \frac{2(x-\cos\alpha)}{3\sin^4\alpha(x^2-2x\cos\alpha+1)^{\frac{1}{2}}}.$$

64. $\int \frac{dx}{(x^3-1)^2}$,

$$\frac{1}{9}\left[\frac{x-1}{x^2+x+1} - \frac{1}{x-1} + \log\frac{x^2+x+1}{(x-1)^2} + 2\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right].$$

65. Derive a formula of reduction for $\int \frac{(a + bx^2)^n}{x^m} dx$.

$$\int \frac{(a + bx^2)^n}{x^m} dx = -\frac{(a + bx^2)^n}{(m - 1)x^{m-1}} + \frac{2nb}{m - 1} \int \frac{(a + bx^2)^{n-1}}{x^{m-2}} dx.$$

66. Derive a formula of reduction for $\int \frac{dx}{x^m \sqrt{(a + bx^2)}}$.

$$\int \frac{dx}{x^m \sqrt{(a + bx^2)}} = \frac{-\sqrt{(a + bx^2)}}{(m - 1)ax^{m-1}} - \frac{(m - 2)b}{(m - 1)a} \int \frac{dx}{x^{m-2} \sqrt{(a + bx^2)}}.$$

67. Develop $\int_0 \frac{\sin x}{x} dx$ in the form of a series.

$$\int_0 \frac{\sin x}{x} dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \ldots$$

VII.

The Integral and its Limits.

81. Before proceeding to some formulæ of integration involving special values of the limits, it will be convenient to resume the consideration of the integral as defined in the first section.

In Art. 3 we supposed a variable magnitude of which the values depend upon some independent variable x (but of which the expression as a function of x is as yet unknown) to vary simultaneously with x, while x passes at a uniform rate over a certain range of values. And it is assumed that the rate of the function is then expressed in terms of x and its assumed rate; or, what is the same thing, that the relative rate of the function as compared with that of x is known. Denoting this relative rate by $f(x)$, a known function of x, let $F(x)$ denote the function under consideration; then by the notation introduced

$$F(x) = \int f(x)\, dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$F(x)$ may now be defined as the function whose derivative is $f(x)$, or whose differential is $f(x)dx$; hence, from this point of view, integration is the search for the indefinite integral, or the inverse of the process of finding the derivative of a function, equation (1) implying nothing more than that

$$d\{F(x)\} = f(x)\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But because this equation was found to be insufficient to fix the values of the function F, we introduced in Art. 4 the notation of the subscript or lower limit; so that, when we write

$$F(x) = \int_a f(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we fully define the value of $F(x)$ by implying the additional condition that $F(a) = 0$. The integral thus modified is sometimes called a *corrected integral*, because the indefiniteness arising from the unknown constant of integration has been removed. It remains "indefinite" in the sense that it is a function of a variable x to which no special value has been assigned. It is to be noticed that, in many applications, the constant is determined (and the integral thus "corrected") by the condition that some other value (not zero) of $F(x)$ shall correspond to a given value of x. These given simultaneous values of x and $F(x)$ are called *the initial values.*

82. As explained in Art. 4, the value of x to which corresponds the required value of the magnitude is known as the *final value* of x, and is used as the upper limit of the integral. Denoting it by b, we thus assume that, as the independent variable x passes from a to b, the function $F(x)$ passes *by continuous variation* from its initial to its final value; and, when this is the case, we may write

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad . \quad . \quad . \quad . \quad (4)$$

in which the function F is defined, not necessarily by equation (3), which implies $F(a) = 0$, but by the more general equation (1), or simply by equation (2).

We may now define the definite integral in equation (4) as *the increment received by a variable whose differential is* $f(x)\,dx$ *while* x *passes from the value* a *to the value* b; or, as *the magnitude generated at the rate* $f(x)\frac{dx}{dt}$ *while* x, *with the arbitrary rate* $\frac{dx}{dt}$, *varies from* a *to* b.

The condition mentioned above, that $F(x)$ shall vary continuously, requires that, starting from some finite initial value $F(a)$, *it shall not become infinite or imaginary for any value of* x *between* a *and* b. The function $F(x)$ cannot become infinite or imaginary unless its derivative $f(x)$ becomes infinite or imaginary. Hence the condition will be fulfilled if $f(x)$ remains real and finite for all such values of x. It is, of course, also assumed that there is no ambiguity about the value of $f(x)$ corresponding to any of these values of x. Cases in which such ambiguity might arise will be considered later.

83. Since the rate of x is arbitrary, we may regard dx in the expression for the integral as constant, and in that case it must be regarded as positive or negative according as b is greater or less than a; in other words, dx must have the sign of $b - a$. When $f(x)$ does not change sign for any value between a and b, we can thus infer the algebraic sign of the definite integral.

Thus, if $b > a$ and $f(x)$ is positive for all values between the limits, $F(x)$ is an increasing function of x, and x is itself increasing; therefore the definite integral denotes a positive increment. For example,

$$\int_0^\pi \frac{\sin x}{x}\,dx$$

denotes a positive quantity. Moreover, since in this case $f(x)$

is less than unity for all values between the limits, the integral is obviously less than the increment received by x, that is, less than π.

It is evident from these considerations that an interchange of the limits changes the sign of the integral; thus,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

which agrees also with equation (4). Again, we infer from the same equation that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

if c is between the limits a and b; and this is also true when c is outside of these limits, provided the condition mentioned in the preceding article holds for the entire range of values of x implied by the several integrals.

84. Returning now to the corrected integral, equation (3), we see that it is the same thing as

$$\int_a^x f(x)\,dx,$$

in which x stands at once for the upper limit and for the independent variable which, in the generation of the integral, is conceived to vary from the initial to the final value. When there is any danger of confusion between these two meanings of x, it is well to use some other letter for the independent variable; thus

$$\int_a^x f(z)\,dz$$

has the same meaning as the expression above; and in general it must be remembered that *an integral is a function of its limits*, and not of the variable which appears under the integral sign unless the same letter serves also to represent one of the limits.

Graphic Representation of an Integral.

85. A geometrical illustration was given in Art. 3, in which $f(x)$ was taken as the ordinate of a curve, and the integral was in consequence represented by an area. We shall now employ another illustration, in which the ordinate y represents the integral itself, regarded as a function of its upper limit. In other words, we shall consider what is called *the graph of the function* (or graphic representation, employing rectangular coordinates), and shall regard this curve as derived from the expression for the integral, and not from the result of any process of integration. Putting

$$y = \int_a f(x)\,dx, \quad \ldots\ldots\ldots \quad (1)$$

we have

$$\frac{dy}{dx} = f(x) = \tan\phi, \quad \ldots\ldots\ldots \quad (2)$$

where ϕ has the same meaning as in Diff. Calc., Art. 38.

Thus for every value of x we know the inclination ϕ of the curve to the axis of x. The notation also implies that when $x = a$, $y = 0$. Starting, then, from the position $(a, 0)$, if we imagine the point (x, y) always to move in the proper direction, a direction which changes as x changes, it will trace out a definite curve, so that the function y has a definite value for each value of x.

86. As an example, let us take the function

$$y = \int_0 \frac{\sin x}{x} dx, \quad \ldots\ldots\ldots \quad (1)$$

a case in which the indefinite integration cannot be performed. Starting from the origin, the describing point must move in the varying direction defined by

$$\tan\phi = \frac{\sin x}{x}. \quad \ldots\ldots\ldots \quad (2)$$

When $x = 0$, we have $\tan \phi = 1$, or $\phi = 45°$; the curve therefore starts from the origin at this inclination. But, since equation (2) shows that $\tan \phi$ decreases as x increases, the curve as we proceed toward the right lies below the tangent at the origin, as in Fig. 3. The ordinate y continues, however, to increase (compare Art. 83) until, at the point A, x reaches the value π, for which $\phi = 0$. As x passes through this value, ϕ changes sign; therefore y has reached a maximum value. From $x = \pi$ to $x = 2\pi$ $\tan \phi$ is negative, hence y decreases; in this interval $\sin x$ in equation (2) goes through numerically the same values as before, but the denominator x is now much larger than before, hence it is plain that y does not decrease to zero. In like manner it is obvious that it will increase again from $x = 2\pi$ to $x = 3\pi$, and so on, reaching alternate maxima and minima, which continually approach an asymptotic value of y corresponding to $x = \infty$. The form of the curve for positive values of x is therefore that represented in Fig. 3. For negative values of x the curve has a similar branch in the third quadrant. It is evident that we can in no case have a finite value of the integral when $x = \infty$, unless, as in this case, the quantity under the integral sign approaches zero as a limit when x increases without limit.

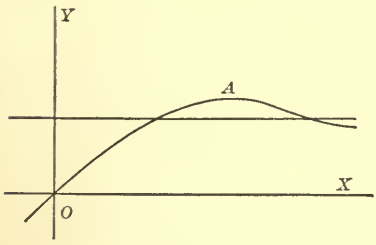

FIG. 3.

87. When the graph or curve of the indefinite integral is drawn, the definite integral between any limits c and d is represented by the increment of y in passing from the abscissa c to the abscissa d, and the condition given in Art. 82 requires that the curve shall be continuous between the points corresponding to the limits; in other words, that *these points should belong to the same branch of the curve.*

In the illustration above, $f(x)$ remains real and finite for *all*

values of x; accordingly, in this case, all values of the limits are possible. But, if $f(x)$ becomes infinite or imaginary for any value of x between the limits, it will usually be found that, although the indefinite integral $y = F(x)$ may have a finite value at each limit, the corresponding points will belong to separate branches of the curve, and therefore equation (4) of Art. 82 cannot be used.

For example, if $f(x) = x^{-2}$, which is infinite for $x = 0$, we have for the indefinite integral

$$y = \int \frac{dx}{x^2} = -\frac{1}{x},$$

and y is also infinite when $x = 0$.

The graph of this integral is an equilateral hyperbola, by means of which we can represent, for example, the definite integral between the limits -2 and -1 as the difference BR between two values of the ordinate. But we cannot interpret in this case an integral between whose limits the value $x = 0$ lies (that is, with one negative and one positive limit), because the corresponding points would lie on disconnected branches of the curve.

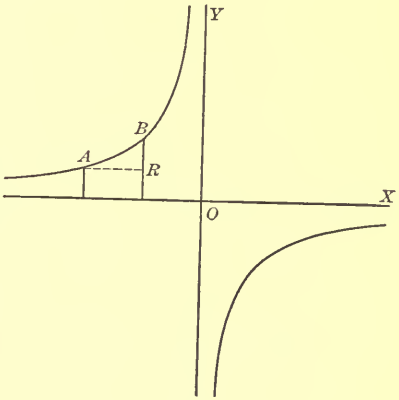

FIG. 4.

It will be noticed that, for the same reason, if we use the notation of the corrected integral we cannot represent the entire curve by a single equation; thus the two branches as drawn in Fig. 4 have the separate equations

$$y = \int_{-\infty}^{x} \frac{dx}{x^2} \quad \text{and} \quad y = \int_{\infty}^{x} \frac{dx}{x^2}.$$

the latter expression denoting an essentially negative quantity.

88. It may happen, however, that the indefinite integral remains finite for a value of x which makes $f(x)$ infinite. In such a case, if the value of $f(x)$ remains real while x passes through this value, we shall still have a continuous curve and a continuous variation of the ordinate between limits which include this value of x. For example, when $f(x) = x^{-\frac{1}{3}}$, which is infinite for $x = 0$, and is real for both positive and negative values, we have

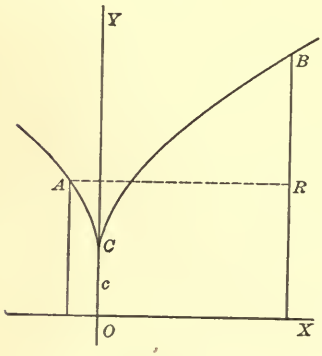

FIG. 5.

$$y = \int \frac{dx}{x^{\frac{1}{3}}} = \frac{3}{2} x^{\frac{2}{3}} + c.$$

The curve which is drawn in Fig. 5 touches the axis of y and is real on both sides of it, forming a cusp. Thus the ordinate varies continuously while x passes through zero, and we can write, for instance,

$$\int_{-1}^{8} \frac{dx}{x^{\frac{1}{3}}} = \frac{3}{2} x^{\frac{2}{3}} \Big]_{-1}^{8} = 6 - 1\tfrac{1}{2} = 4\tfrac{1}{2},$$

in accordance with equation (4), Art. 82, as illustrated in Fig. 5 by the difference of ordinates BR.

Multiple-valued Integrals.

89. When the indefinite integral is a many-valued function, the condition that it shall vary continuously while x passes from the lower to the upper limit requires the selection of properly corresponding values at the two limits. In illustra-

tion let us construct the graph of the fundamental integral

$$y = \int_0 \frac{dx}{\sqrt{(1 - x^2)}} = \sin^{-1} x. \quad . \quad . \quad . \quad . \quad (1)$$

In this case x can only vary between the values -1 and $+1$. Proceeding from the origin, the curve, Fig. 6, has the inclination 45°, which is gradually increased to 90° when $x = 1$, at the point A, for which the value of y is $\frac{1}{2}\pi$. As x passes from 0 to -1 a similar branch is described in the third quadrant, completing the curve drawn in full line, the ordinate of which is in fact *the primary value* of the indefinite integral $\sin^{-1} x$ (Diff. Calc., Art. 74). But the complete curve $y = \sin^{-1} x$, or $x = \sin y$, has another branch continuous with this at the point $(1, \frac{1}{2}\pi)$, as represented by the dotted line. In order that the point moving in accordance with the integral expression shall describe this branch, we must suppose that x, after reaching the value unity at the point A, begins to decrease,

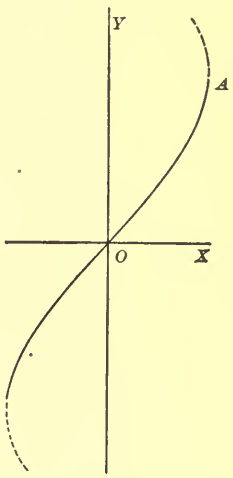

FIG. 6.

and that, as dx thus changes sign, the radical $\sqrt{(1 - x^2)}$, which then becomes zero, also changes sign, so that the quantity under the integral sign remains positive, and y goes on increasing. Since the radical cannot change sign, while varying continuously, except when it passes through the value zero (at which time its two values are equal), it is only when $x = \pm 1$ that it can return to its previous values without causing y also to return to the corresponding previous values; that is, without making the generating point return upon the path already described.

We may thus, by the alternate increase and decrease of x between its extreme values, describe an infinite number of

branches. In this generation of the integral the value of the radical undergoes periodic change of sign but is never ambiguous; for, in drawing the curve we assumed that in equation (1) the radical had its positive value when $x = 0$ and $y = 0$. This assumption determines the value of the radical for every other value of y; it is, in fact, always equal to $\cos y$.

90. In the direct application of limits to the integral (1), it is of course sufficient to limit the indefinite integral to its primary value, the radical being positive for all values of x between the limits; but, when the integral is the result of transformation, care may become necessary in the selection of the values at the limits.

Consider, for example, the integral

$$\int_{-1}^{\sqrt{2}} \frac{dz}{\sqrt{(2 - z^2)}},$$

which can be evaluated directly by formula (f') in the form $\sin^{-1} \frac{z}{\sqrt{2}} \Big]_{-1}^{\sqrt{2}} = \frac{3\pi}{4}$. But suppose we make the transformation $z^2 = 1 - x$, whence $dz = \frac{-dx}{2z}$, and $\sqrt{(2 - z^2)} = \sqrt{(1 + x)}$.

It is first to be noticed that, because the value of z is negative at the lower limit, we should in the value of dz put $z = -\sqrt{(1 - x)}$; thus $dz = \frac{dx}{2\sqrt{(1 - x)}}$, and the result of transformation is

$$\int_{-1}^{\sqrt{2}} \frac{dz}{\sqrt{(2 - z^2)}} = \frac{1}{2} \int_{0}^{-1} \frac{dx}{\sqrt{(1 - x^2)}}.$$

The relation $z^2 = 1 - x$ shows that as z increases from -1 to 0, x increases from 0 to 1, and the corrected integral in the second member (which is $\sin^{-1} x$) increases from 0 to $\frac{1}{2}\pi$. Then as z further increases from 0 to $\sqrt{2}$, x decreases from 1 to -1; but the integral continues to increase from $\frac{1}{2}\pi$ through

values which are not "primary," reaching the final value $\frac{3}{2}\pi$ at the upper limit, so that we have for the value of the second member of the equation $\frac{1}{2}(\frac{3}{2}\pi - 0) = \frac{3}{4}\pi$ as before.

91. To illustrate another mode in which an integral may acquire multiple-values, let us consider the graph of the fundamental integral

$$y = \int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} x.$$

Proceeding from the origin at the inclination 45°, the curve approaches, as x increases without limit, an asymptote parallel to the axis of x at the distance $\frac{1}{2}\pi$. The similar branch described for negative values of x completes the full line in Fig. 7, representing the primary values of $\tan^{-1} x$, which are employed in any direct application of limits to the integral. But the complete curve $y = \tan^{-1} x$ consists of an unlimited number of branches which are repetitions of this curve at successive vertical intervals each equal to π, so that each asymptote is approached by two branches.

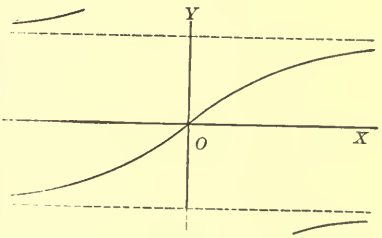

FIG. 7.

Now, when the integral arises from transformation, it may happen that x passes through infinity, changing sign, and then the ordinate will pass without discontinuity of value from one branch to the next. For example, given the integral

$$\int_0^{\frac{5\pi}{6}} \frac{d\theta}{\cos^2\theta + 9\sin^2\theta} = \int_0^{\frac{5\pi}{6}} \frac{\sec^2\theta\, d\theta}{1 + 9\tan^2\theta};$$

if we put $\tan\theta = x$, this becomes

$$\int_0^{-\frac{\sqrt{3}}{3}} \frac{dx}{1 + 9x^2} = \frac{1}{3}\tan^{-1} 3x \Big]_0^{-\frac{\sqrt{3}}{3}} = \frac{1}{3}[\tan^{-1}(-\sqrt{3}) - \tan^{-1} 0].$$

As θ passes from 0 to $\frac{1}{2}\pi$, x increases from 0 to $+\infty$, and $\tan^{-1} 3x$ increases from 0 to $\frac{1}{2}\pi$. But as θ further increases from $\frac{1}{2}\pi$ to $\frac{5}{6}\pi$, x changes sign and passes from $-\infty$ to $-\frac{1}{3}\sqrt{3}$. During this interval $\tan^{-1} 3x$ still further increases from $\frac{1}{2}\pi$ to $\frac{8}{6}\pi$, which is therefore in this case to be taken as the value of $\tan^{-1}(-\sqrt{3})$, while we take zero as the value of $\tan^{-1} 0$. Hence

$$\int_0^{\frac{5\pi}{6}} \frac{d\theta}{\cos^2\theta + 9\sin^2\theta} = \frac{2\pi}{9}.$$

Formulæ of Reduction for Definite Integrals.

92. The limits of a definite integral are very often such as to simplify materially the formula of reduction appropriate to it. For example, to reduce

$$\int_0^\infty x^n e^{-x} dx,$$

we have by the method of parts

$$\int x^n e^{-x} dx = -x^n e^{-x} + n\int e^{-x} x^{n-1} dx.$$

Now, supposing n positive, the quantity $x^n e^{-x}$ vanishes when $x = 0$, and also when $x = \infty$. [See Diff. Calc., Art. 159.] Hence, applying the limits 0 and ∞,

$$\int_0^\infty x^n e^{-x} dx = n\int_0^\infty x^{n-1} e^{-x} dx.$$

By successive application of this formula we have, when n is an integer,

$$\int_0^\infty x^n e^{-x} dx = n(n-1)\cdots\cdots\cdots 2\cdot 1 = n!$$

93. From equation (1), Art. 66, supposing $m > 1$, we have

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \, d\theta.$$

If m is an integer, we shall, by successive application of this formula, finally arrive at $\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$ or $\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 1$, according as m is even or odd. Hence

if m is even, $$\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta = \frac{(m-1)(m-3)\cdots\cdot 1}{m(m-2)\cdots\cdots 2} \cdot \frac{\pi}{2}, \quad \ldots (P)$$

and if m is odd, $$\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta = \frac{(m-1)(m-3)\cdots\cdot 2}{m(m-2)\cdots\cdots 1}. \quad \ldots (P')$$

94. From equations (3) and (4) Art. 69, we derive

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^{n-2} \theta \, d\theta,$$

and $$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta \, d\theta.$$

By successive applications of the first of these formulæ, we produce a series of factors in the numerator decreasing by intervals of two units from $n - 1$, and ending with 1 or 2, as in formulæ (P) and (P'). By the second formula we then produce a like series of factors in the numerator beginning with $m - 1$. The successive factors in the denominator produced in the process form a single series beginning with $m + n$ and decreasing by intervals of 2, the final factor being greater by 2 than the sum of the exponents in the final integral. Now this final integral will take one of the four forms

$$\int_0^{\frac{\pi}{2}} d\theta, \qquad \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta, \qquad \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta, \qquad \text{or} \qquad \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta,$$

according as m and n are even or odd. Thus the final factors in the denominator are, in the four cases respectively,

$$2, \qquad 3, \qquad 3, \qquad 4;$$

while the values of the final integrals are respectively

$$\tfrac{1}{2}\pi, \qquad 1, \qquad 1, \qquad \tfrac{1}{2}.$$

The result will therefore be the same if we carry the series of factors in the denominator in all cases to 1 or 2, and ignore the final integral except in the first case (m and n both even), when its value is $\frac{1}{2}\pi$. Therefore we may write, when m and n are positive integers,

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots}\alpha, \quad (Q)$$

provided that each series of factors is carried to 2 or 1, *and* α *is taken equal to unity, except when* m *and* n *are both even, in which case* $\alpha = \frac{1}{2}\pi$.

This formula, of course, includes formulæ (P) and (P').

Change of Independent Variable in a Definite Integral.

95. A change of independent variable is frequently made for the purpose of simplifying the limits, rather than of changing the form of the integral. For example, suppose the limits in formula (Q) to be multiples of $\frac{1}{2}\pi$ by any consecutive integers k and $k+1$. Putting $\theta = \frac{1}{2}k\pi + \phi$, we have, according as k is even or odd,

$$\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \pm \int_0^{\frac{\pi}{2}} \sin^m\phi \cos^n\phi\, d\phi,$$

or

$$\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \pm \int_0^{\frac{\pi}{2}} \cos^m\phi \sin^n\phi\, d\phi.$$

But by formula (Q) the integrals in the second members have the same value; hence *the numerical value of the integral of formula* (Q) *is the same for every quadrant.* The sign to be used is obviously that of $\sin^m\theta \cos^n\theta$ in the quadrant in question. We can thus readily determine the value of the integral when the limits are any multiples of $\frac{1}{2}\pi$. When m and n are both even, the sign is always $+$, and we have to multiply the result of formula (Q) by the number of quadrants. In other cases, we have two positive and two negative results while θ describes one revolution.

96. When the limits of an integral of the form considered above are 0 and $\frac{1}{4}\pi$, we may by introducing the double angle obtain the limits 0 and $\frac{1}{2}\pi$. Since, if $\phi = 2\theta$,

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin\phi, \quad \sin^2\theta = \tfrac{1}{2}(1-\cos\phi), \quad \cos^2\theta = \tfrac{1}{2}(1+\cos\phi),$$

the integral will, *whenever* m *and* n *are either both even or both odd,* be thus converted into one or more integrals of the same

form, with integral values of the exponents, and having limits of the desired form.

For example, by this transformation we obtain

$$\int_0^{\frac{\pi}{4}} \sin^6\theta \cos^4\theta\, d\theta = \frac{1}{64}\int_0^{\frac{\pi}{2}} \sin^4\phi\,(1-\cos\phi)\, d\phi$$

$$= \frac{1}{64}\left[\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} - \frac{3\cdot 1\cdot 1}{5\cdot 3\cdot 1}\right] = \frac{1}{64}\left[3\frac{\pi}{16} - \frac{1}{5}\right].$$

97. A transformation which does not change, or which interchanges, the limits may sometimes reproduce the given integral together with known integrals, and thus lead to its evaluation. When the limits a and b of any integral are finite, they will be reversed by the substitution

$$x = a + b - z.$$

Thus

$$\int_a^b f(x)\, dx = -\int_b^a f(a+b-z)\, dz = \int_a^b f(a+b-z)\, dz;$$

or, since it is indifferent whether we write z or x for the independent variable in a definite integral,

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$$

For example,

$$\int_0^\pi \theta \sin^4\theta\, d\theta = \int_0^\pi (\pi-\theta)\sin^4(\pi-\theta)\, d\theta = \int_0^\pi (\pi-\theta)\sin^4\theta\, d\theta;$$

hence

$$2\int_0^\pi \theta \sin^4\theta\, d\theta = \pi\int_0^\pi \sin^4\theta\, d\theta,$$

and, by formula (P),

$$\int_0^\pi \theta \sin^4\theta\, d\theta = \pi\,\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi^2}{16}.$$

98. When the limits are 0 and ∞, they will be reversed if we assume for the new variable the reciprocal, or a multiple of the reciprocal, of x. For example, if in

$$u = \int_0^\infty \frac{\log x}{a^2 + x^2}\, dx$$

we put $x = \frac{a^2}{y}$, whence $dx = -\frac{a^2\, dy}{y^2}$, we find

$$u = \int_0^\infty \frac{2 \log a - \log y}{y^2 + a^2}\, dy = 2 \log a \int_0^\infty \frac{dy}{y^2 + a^2} - u;$$

hence

$$u = \log a \int_0^\infty \frac{dy}{y^2 + a^2} = \frac{\pi \log a}{2a}.$$

The Integral regarded as the Limit of a Sum.

99. We have seen in Art. 3 that, if the curve $y = f(x)$ be constructed, the integral $\int_a^b f(x)\, dx$ (in which we suppose $f(x)$ to remain finite and continuous as x varies from a to b) is represented by the area included by the curve, the axis of x and the ordinates corresponding to $x = a$ and $x = b$.

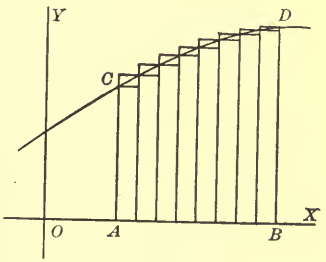

FIG. 8.

Let CD in Fig. 8 be the curve, and let the base of this area, AB, whose length is $b - a$, be divided into n equal parts. Denote the length of each part by Δx, so that

$$b = a + n\, \Delta x,$$

and erect an ordinate as in the figure at each point of division.

The whole area is thus divided into n parts, as represented by the equation

$$\int_a^b y dx = \int_a^{a+\Delta x} y dx + \int_{a+\Delta x}^{a+2\Delta x} y dx + \ldots + \int_{b-\Delta x}^{b} y dx. \quad (1)$$

Now if we denote the values of y corresponding to $a + \Delta x$, $a + 2\Delta x, \ldots b$ by $y_1, y_2, \ldots y_n$, the rectangles $y_1\Delta x$, $y_2\Delta x, \ldots$ $y_n\Delta x$, which are constructed in the figure, are approximations to the several terms of the second member of equation (1). Hence the sum of these rectangles,

$$\Sigma y\Delta x = y_1\Delta x + y_2\Delta x + \ldots + y_n\Delta x, \quad . \quad . \quad . \quad (2)$$

is an approximate expression for the integral in equation (1). When, as in Fig. 8, y increases continuously while x passes from a to b, the sum of the rectangles exceeds the curvilinear area; that is, $\Sigma y\Delta x$ exceeds the integral.

If we take for the altitude of each rectangle the initial instead of the final value of y in the interval, we shall obtain in like manner an approximate expression,

$$\Sigma' y\Delta x = y_0\Delta x + y_1\Delta x + \ldots + y_{n-1}\Delta x, \quad . \quad (3)$$

which will, in this case, represent an area less than the curvilinear one, so that $\Sigma' y\Delta x$ is less than the integral. Thus the integral is intermediate in value to the two expressions in equations (2) and (3), of which the difference is

$$\Sigma y\Delta x - \Sigma' y\Delta x = (y_n - y_0)\Delta x. \quad . \quad . \quad . \quad (4)$$

Therefore the difference between the integral and either of the approximate expressions is *less* than $(y_n - y_0)\Delta x$.

Now when the number of parts n is indefinitely increased,

so that Δx is decreased without limit, the limit of $(y_n - y_0)\,\Delta x$ is zero; it follows that $\int y\,dx$ is the limit of $\Sigma y\,\Delta x$ for any range of values of x for which y is an increasing function of x. It can be shown in like manner that the same thing is true while y is a decreasing function, and therefore in general

$$\int_a^b f(x)\,dx = \text{the limit of } \sum_{x=a}^{x=b} f(x)\,\Delta x \quad . \quad . \quad . \quad (5)$$

when Δx decreases without limit.

100. The typical term $f(x)\,\Delta x$ is called *the element* of the sum in equation (5). It will be noticed that the sum will have the same limit whether the x in $f(x)\,\Delta x$ be taken as in equation (2) or as in equation (3), or in any other manner, *provided it be some value within the interval to which Δx corresponds.* The expression $f(x)\,dx$ is, in like manner, called *the element of the integral* in equation (5).

In many applications of the Integral Calculus, the expression to be integrated is obtained by regarding it as an element. In so doing we are really dealing with the element of the sum; but as we intend to pass to the limit we may, in accordance with the remark made above, ignore the distinction between any values of x between x and $x + \Delta x$. By equation (5), we pass to the limit by simply writing d in place of Δ, and the integral sign in place of that of summation,* so that in practice it is customary to form the element of the integral at once by writing d instead of Δ.

* The term *integral*, and the use of the long *s*, the initial of the word *sum*, as the sign of integration, have their origin in this connection between the processes of integration and summation.

Additional Formulæ of Integration.

101. The formulæ recapitulated below are useful in evaluating other integrals. (*A*) and (*A'*) are demonstrated in Art. 17; (*B*) and (*C*) in Art. 29; (*D*) and (*E*) in Art. 30; (*F*) in Art. 31; (*G*) and (*G'*) in Art. 35; (*H*) and (*I*) in Art. 50; (*J*) in Art. 51; (*K*) in Art. 52; (*L*) in Art. 53; (*M*) in Art. 55; (*N*) and (*O*) in Art. 58; (*P*) and (*P'*) in Art. 93; and (*Q*) in Art. 94.

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b} \log \frac{x-a}{x-b}. \quad \ldots \quad (A)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}; \quad \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}. \quad \ldots \quad (A')$$

$$\int \sin^2\theta \, d\theta = \tfrac{1}{2}(\theta - \sin\theta\cos\theta). \quad \ldots \quad (B)$$

$$\int \cos^2\theta \, d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta). \quad \ldots \quad (C)$$

$$\int \frac{d\theta}{\sin\theta\cos\theta} = \log\tan\theta. \quad \ldots \quad (D)$$

$$\int \frac{d\theta}{\sin\theta} = \log\tan\tfrac{1}{2}\theta = \log\frac{1-\cos\theta}{\sin\theta}. \quad \ldots \quad (E)$$

$$\int \frac{d\theta}{\cos\theta} = \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right] = \log\frac{1+\sin\theta}{\cos\theta} = \log(\sec\theta+\tan\theta). \quad (F)$$

$$\int \frac{d\theta}{a+b\cos\theta} = \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\tfrac{1}{2}\theta\right]. \quad \ldots \quad (G)$$

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{(b+a)} + \sqrt{(b-a)}\tan\frac{1}{2}\theta}{\sqrt{(b+a)} - \sqrt{(b-a)}\tan\frac{1}{2}\theta} \quad . \quad (G')$$

$$\int \frac{dx}{x\sqrt{(x^2 + a^2)}} = \frac{1}{a} \log \frac{\sqrt{(x^2 + a^2)} - a}{x} \quad . \quad (H)$$

$$\int \frac{dx}{x\sqrt{(a^2 - x^2)}} = \frac{1}{a} \log \frac{a - \sqrt{(a^2 - x^2)}}{x} \quad . \quad (I)$$

$$\int \frac{dx}{(ax^2 + b)^{\frac{3}{2}}} = \frac{x}{b\sqrt{(ax^2 + b)}} \quad . \quad (J)$$

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log [x + \sqrt{(x^2 \pm a^2)}] \quad . \quad (K)$$

$$\int \sqrt{(x^2 \pm a^2)}\, dx = \frac{x\sqrt{(x^2 \pm a^2)}}{2} \pm \frac{a^2}{2} \log [x + \sqrt{(x^2 \pm a^2)}] \quad . \quad (L)$$

$$\int \sqrt{(a^2 - x^2)}\, dx = \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{x\sqrt{(a^2 - x^2)}}{2} \quad . \quad (M)$$

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} = 2 \log [\sqrt{(x-\alpha)} + \sqrt{(x-\beta)}] \quad . \quad (N)$$

$$\int \frac{dx}{\sqrt{[(x-\alpha)(\beta - x)]}} = 2 \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} \quad . \quad (O)$$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^m\theta\, d\theta = \frac{(m-1)(m-3)\cdots 1}{m(m-2)\cdots 2} \cdot \frac{\pi}{2} \quad . \quad (P)$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^m \theta \, d\theta = \frac{(m-1)(m-3)\cdots 2}{m(m-2)\cdots 1} \quad \ldots \quad (P')$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3)\cdots \times (n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \alpha, \quad \ldots \quad (Q)$$

in which $\alpha = 1$, unless m and n are both even, when $\alpha = \frac{\pi}{2}$.

Examples VII.

1. $\int_0^{n\pi} \frac{d\theta}{a + b\cos\theta}$, [$a > b$, and n an integer] $\frac{n\pi}{\sqrt{(a^2 - b^2)}}$.

2. $\int_0^{2n\pi \pm \frac{\pi}{2}} \frac{d\theta}{2 + \cos\theta}$, $\frac{2n\pi \pm \frac{1}{3}\pi}{\sqrt{3}}$.

3. $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta$, $\frac{5\pi}{32}$.

4. $\int_0^{\pi} \sin^5 \theta \, d\theta$, $\frac{16}{15}$.

5. $\int_{-\frac{\pi}{2}}^{\pi} \cos^7 \theta \, d\theta$, $\frac{16}{35}$.

6. $\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^6 \theta \, d\theta$, $\frac{3\pi}{512}$.

7. $\int_0^{\pi} \sin^3\theta \cos^4\theta \, d\theta$, $\frac{4}{35}$.

8. $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^m\theta \, d\theta$, $\frac{1}{2^m}\int_0^{\frac{\pi}{2}} \sin^m\theta \, d\theta$.

9. $\int_0^1 \frac{x^{2n}\,dx}{\sqrt{(1-x^2)}}$, $\frac{1\cdot3\cdot5\cdot\cdots(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot2n}\frac{\pi}{2}$.

10. $\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{(1-x^2)}}$. $\frac{2\cdot4\cdot6\cdot\cdots\cdot2n}{3\cdot5\cdot7\cdot\cdots(2n+1)}$.

11. $\int_0^a x^3(a^2-x^2)^{\frac{5}{2}}\,dx$, $\frac{2a^9}{63}$.

12. $\int_a^{\infty} \frac{(x^2-a^2)^{\frac{3}{2}}\,dx}{x^5}$, $\frac{3\pi}{16a}$.

13. $\int_0^{\infty} \frac{x^5\,dx}{(a^2+x^2)^{\frac{7}{2}}}$, $\frac{8}{15a}$.

14. $\int_0^{\infty} \frac{x^4\,dx}{(a^2+x^2)^4}$, $\frac{\pi}{32a^3}$.

15. Prove that

$$\int_0^a x^{n-1}(a-x)^{m-1}\,dx = \int_0^a x^{m-1}(a-x)^{n-1}\,dx,$$

and derive a formula of reduction for this integral, supposing $n > 0$ and $m > 1$.

$$\int_0^a x^{n-1}(a-x)^{m-1}\,dx = \frac{m-1}{n}\int_0^a x^n(a-x)^{m-2}\,dx.$$

16. Deduce from the result of Ex. 15 the value of the integral when m is an integer.

$$\int_0^a x^{n-1}(a-x)^{m-1}\,dx = \frac{1\cdot 2\cdot 3\cdot\cdots(m-1)}{n(n+1)\cdots(n+m-1)}a^{m+n-1}.$$

17. $\int_{-a}^{a}(a+x)^5(a-x)^{\frac{3}{2}}\,dx$. *See Ex.* 16. $\frac{2^{16}\sqrt{2}}{45045}a^{\frac{15}{2}}$.

18. $\int_0^{\frac{\pi}{2}}\sin^7\theta\,(\cos\theta)^{\frac{5}{2}}\,d\theta$. *Put* $\sin^2\theta = x$, *and see Ex.* 16. $\frac{2^8}{5\cdot 7\cdot 11\cdot 19}$.

19. $\int_0^{\frac{\pi}{4}}\cos^8\theta\,d\theta$, $\frac{35\pi}{256}+\frac{5}{24}$.

20. $\int_0^{\frac{\pi}{4}}\sin^6\theta\cos^2\theta\,d\theta$, $\frac{5}{384}$.

21. $\int_0^a \frac{x^4\,dx}{(a^2+x^2)^4}$, $\frac{3\pi-4}{192a^3}$.

22. Evaluate $\int_0^{\pi}\theta\cos^2\theta\,d\theta$ (see Ex. VI, 48) and thence derive $\int_0^{\pi}\theta^2\cos^2\theta\,d\theta$ by the method of Art. 97. $-\frac{14\pi}{9}$.

CHAPTER III.

Geometrical Applications—Double and Triple Integrals.

VIII.

Plane Areas.

102. The first step in making an application of the Integral Calculus is to express the required magnitude in the form of an integral. In the geometrical applications, the magnitude is regarded as generated while some selected independent variable undergoes a given change of value. The independent variable is usually a straight line or an angle, varying between known limits; the required magnitude is either a line regarded as generated by the motion of a point, an area generated by the motion of a line, or a solid generated by the motion of an area. A plane area may be generated by the motion of a straight line, generally of variable length, the method selected depending upon the mode in which the boundaries of the area are defined.

The Area generated by a Variable Line having a Fixed Direction.

103. The differential of the area generated by the ordinate of a curve, whose equation is given in rectangular coordinates, has been derived in Art. 3. The same method may be employed in the case of any area generated by a straight line whose direction is invariable.

Let AB be the generating line, and let R be its intersection with a fixed line CD, to which it is always perpendicular. Suppose R to move uniformly along CD, and let RS be the space described by R in the interval of time, dt. Then the value of the differential of the area, at the instant when the generating line passes the position AB, is the area which would be generated in the time dt, if the rate of the area were constant. This rate would evidently become constant if the generating line were made constant in length; and therefore the differential is the rectangle, represented in the figure, whose base and altitude are AB and RS; that is, it is *the product of the generating line, and the differential of its motion in a direction perpendicular to its length.*

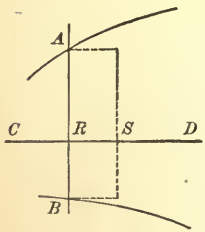

FIG. 9.

104. In the algebraic expression of this principle, the independent variable is the distance of R from some fixed origin upon CD, and the length of AB is to be expressed in terms of this independent variable.

When the curve or curves defining the length of AB are given in rectangular coordinates, CD is generally one of the axes; thus, if the generating line is the ordinate of a curve, the differential is $y\,dx$, as shown in Art. 3. It is often, however, convenient to regard the area as generated by some other line.

For example, given the curve known as the witch, whose equation is

$$y^2x - 2ay^2 + 4a^2x = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This curve passes through the origin, is symmetrical to the axis of x, and has the line $x = 2a$ for an asymptote, since $x = 2a$ makes $y = \pm\infty$.

Let the area between the curve and its asymptote be re-

quired. We may regard this area as generated by the line PQ parallel to the axis of x, y being taken as the independent variable. Now

$$PQ = 2a - x,$$

hence the required area is

$$A = \int_{-\infty}^{\infty} (2a - x)\, dy \quad . \quad . \quad . \quad . \quad (2)$$

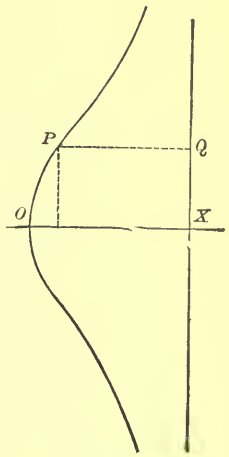

FIG. 10.

From the equation (1) of the curve, we have

$$x = \frac{2ay^2}{y^2 + 4a^2};$$

whence $$2a - x = \frac{8a^3}{y^2 + 4a^2},$$

and equation (2) becomes

$$A = 8a^3 \int_{-\infty}^{\infty} \frac{dy}{y^2 + 4a^2} = 4a^2 \tan^{-1} \frac{y}{2a} \Big]_{-\infty}^{\infty} = 4\pi a^2.$$

Oblique Coordinates.

105. When the coordinate axes are oblique, if α denotes the angle between them, and the ordinate is the generating line, the differential of its motion in a direction perpendicular to its length is evidently $\sin \alpha \cdot dx$; therefore, the expression for the area is

$$A = \sin \alpha \int y\, dx.$$

As an illustration let the area between a parabola and a chord passing through the focus be required. It is shown in treatises on conic sections, that the value of a focal chord is

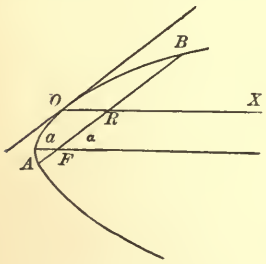

FIG. 11.

$$AB = 4a \operatorname{cosec}^2 \alpha, \quad . \quad . \quad . \quad (1)$$

where α is the inclination of the chord to the axis of the curve, and a is the distance from the focus to the vertex. It is also shown that the equation of the curve referred to the diameter which bisects the chord, and the tangent at its extremity which is parallel to the chord is

$$y^2 = 4a \operatorname{cosec}^2 \alpha \cdot x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The required area may be generated by the double ordinate in this equation; and since from (1) the final value of y is $\pm 2a \operatorname{cosec}^2 \alpha$, equation (2) gives for the final value of x

$$OR = a \operatorname{cosec}^2 \alpha.$$

Hence we have

$$A = 2 \sin \alpha \int_0^{a \operatorname{cosec}^2 \alpha} y \, dx,$$

or by equation (2)

$$A = 4\sqrt{a} \int_0^{a \operatorname{cosec}^2 \alpha} \sqrt{x} \, dx = \frac{8a^2 \operatorname{cosec}^3 \alpha}{3}.$$

Employment of an Auxiliary Variable.

106. We have hitherto assumed that, in the expression

$$A = \int y \, dx,$$

x is taken as the independent variable, so that dx may be assumed constant; and it is usual to take the limits in such a manner that dx is positive. The resulting value of A will then have the sign of y, and will change sign if y changes sign.

It is frequently desirable, however, as in the illustration given below, to express both y and dx in terms of some other variable. When this is done, it is to be noticed that it is not necessary that dx should retain the same sign throughout the entire integral. The limits may often be so taken that the extremityof the generating ordinate must pass completely around a closed curve, and in that case it is easily seen that the complete integral, which represents the algebraic sum of the areas generated positively and negatively, will be the whole area of the closed curve.

107. As an illustration, let the whole area of the closed curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1,$$

represented in Fig. 12, be required. If in this equation we put

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} = \sin\psi,$$

we shall have

$$\left(\frac{y}{b}\right)^{\frac{1}{3}} = \cos\psi;$$

whence $x = a\sin^3\psi$, and $y = b\cos^3\psi$. . . (1)

Therefore $\int y\,dx = 3ab\int \cos^4\psi\sin^2\psi\,d\psi.$

Now if in this integral we use the limits 0 and 2π, the point determined by equation (1) describes the whole curve in the direction $ABCDA$. Hence we have for the whole area

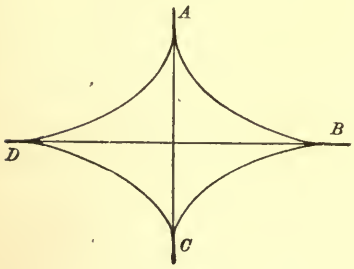

FIG. 12.

$$A = 3ab\int_0^{2\pi} \cos^4\psi \sin^2\psi \, d\psi,$$

and by formula (Q)

$$A = 3ab\,\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\,2\pi = \frac{3\pi ab}{8}.$$

The areas in this case are all generated with the positive sign, since when y is negative dx is also negative. Had the generating point moved about the curve in the opposite direction, the result would have been negative.

Area generated by a Rotating Line or Radius Vector.

108. The radius vector of a curve given in polar coordinates is a variable line rotating about a fixed extremity. The angular rate is denoted by $\frac{d\theta}{dt}$ and may be regarded as constant, but then the rate at which area is generated by the radius vector OP, Fig. 13, will not be constant, since the length OP is not constant. The differential of this area is the area which would be generated in the time dt, if the rate of the area were constant; that is to say, if the radius vector were of constant

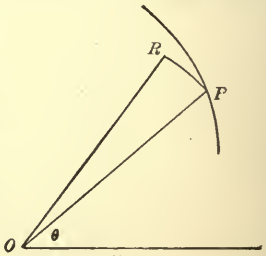

FIG. 13.

length. It is therefore the circular sector OPR of which the radius is r and the angle at the centre is $d\theta$.

Since $$\text{arc } PR = r\,d\theta,$$

$$\text{sector } OPR = \frac{1}{2} r^2\,d\theta;$$

therefore the expression for the generated area is

$$A = \frac{1}{2}\int r^2\,d\theta \quad \ldots\ldots\ldots \quad (1)$$

109. As an illustration, let us find the area of the right-hand loop of the lemniscate

$$r^2 = a^2 \cos 2\theta.$$

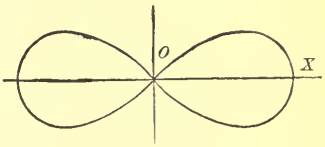

FIG. 14.

The limits to be employed are those values of θ which make $r = 0$; that is $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.
Hence the area of the loop is

$$A = \frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\,d\theta = \frac{a^2}{4}\sin 2\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

110. When the radii vectores, r_2 and r_1 corresponding to the same value of θ in two curves, have the same sign, the area generated by their difference is the difference of the polar areas generated by r_1 and r_2. Hence the expression for this area is

$$A = \frac{1}{2}\int (r_2^2 - r_1^2)\,d\theta. \quad \ldots\ldots \quad (2)$$

111. Let us apply this formula to find the whole area between the cissoid

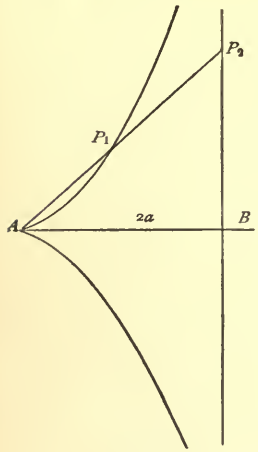

FIG. 15.

$$r_1 = 2a\,(\sec\theta - \cos\theta),$$

Fig. 15, and its asymptote BP_2, whose polar equation is

$$r_2 = 2a\sec\theta.$$

One half of the required area is generated by the line P_1P_2, while θ varies from 0 to $\frac{1}{2}\pi$. Hence by the formula

$$A = 2a^2\int_0^{\frac{\pi}{2}} (2-\cos^2\theta)\,d\theta = \frac{3}{2}\pi a^2.$$

Therefore the whole area required is $3\pi a^2$.

Transformation of the Polar Formulae.

112. In the case of curves given in rectangular coordinates, it is sometimes convenient to regard an area as generated by a radius vector, and to use the transformations deduced below in place of the polar formulæ.

Put $$y = mx; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

now taking the origin as pole and the initial line as the axis of x, we have

$$x = r\cos\theta, \qquad y = r\sin\theta; \quad . \quad . \quad . \quad (2)$$

therefore $$m = \frac{y}{x} = \tan\theta,$$

and $$dm = \sec^2\theta\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From equations (2) and (3),

$$x^2\,dm = r^2\,d\theta;$$

therefore equation (1) of Art. 108 gives

$$A = \frac{1}{2}\int x^2\,dm. \quad \ldots\ldots \quad (4)$$

In like manner, equation (2) Art. 110 becomes

$$A = \frac{1}{2}\int (x_2^2 - x_1^2)\,dm. \quad \ldots\ldots \quad (5)$$

113. As an illustration, let us take the folium

$$x^3 + y^3 - 3axy = 0. \quad \ldots\ldots \quad (1)$$

Putting $y = mx$, we have

$$x^3(1 + m^3) - 3amx^2 = 0. \quad \ldots\ldots \quad (2)$$

This equation gives three roots or values of x, of which two are always equal zero, and the third is

$$x = \frac{3am}{1 + m^3}; \quad \ldots\ldots \quad (4)$$

whence

$$y = \frac{3am^2}{1 + m^3}. \quad \ldots\ldots \quad (5)$$

These are therefore the coordinates of the point P in Fig. 16. Since the values of x and y vanish when $m = 0$, and when $m = \infty$, the curve has a loop in the first quadrant. To find

the area of this loop we therefore have, by equation (4) of the preceding article,

$$A = \frac{9a^2}{2}\int_0^\infty \frac{m^2\,dm}{(1+m^3)^2} = -\frac{3a^2}{2}\frac{1}{1+m^3}\Big]_0^\infty = \frac{3a^2}{2}.$$

114. The area included between this curve and its asymptote may be found by means of equation (5), Art. 112. The equation of a straight line is of the form

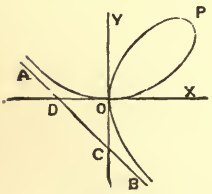

FIG. 16.

$$y = mx + b,$$

and since this line is parallel to $y = mx$, the value of m for the asymptote must be that which makes x and y in equations (4) and (5) infinite; that is, $m = -1$; hence the equation of the asymptote is

$$y + x = b, \quad \ldots\ldots\ldots \quad (6)$$

in which b is to be determined. Since when $m = -1$, the point P of the curve approaches indefinitely near to the asymptote, equation (6) must be satisfied by P when $m = -1$. From (4) and (5) we derive

$$y + x = 3a\frac{m^2+m}{1+m^3} = \frac{3am}{1-m+m^2};$$

whence, putting $m = -1$, and substituting in equation (6)

$$-a = b,$$

the equation of the asymptote AB, Fig. 16, is

$$y + x = -a. \quad \ldots\ldots\ldots \quad (7)$$

Now, as m varies from $-\infty$ to 0, the difference between the radii vectores of the asymptote and curve will generate the areas OBC and ODA, hence the sum of these areas is represented by

$$A = \frac{1}{2}\int_{-\infty}^{0} (x_2^2 - x_1^2)\, dm,$$

in which x_2 is taken from the equation of the asymptote (7), and x_1 from that of the curve.

Putting $y = mx$, in (7), we have

$$x_2 = -\frac{a}{1+m},$$

and the value of x_1 is given in equation (4). Hence

$$A = \frac{a^2}{2}\int_{-\infty}^{0}\left[\frac{1}{(1+m)^2} - \frac{9m^2}{(1+m^3)^2}\right] dm$$

$$= \frac{a^2}{2}\left[\frac{3}{1+m^3} - \frac{1}{1+m}\right]_{-\infty}^{0}$$

$$= \frac{a^2}{2}\left[\frac{2+m-m^2}{1+m^3}\right]_{-\infty}^{0} = \frac{a^2}{2}\left[\frac{2-m}{1-m+m^2}\right]_{-\infty}^{0} * = a^2.$$

Adding the triangle OCD, whose area is $\frac{1}{2}a^2$, we have for the whole area required $\frac{3}{2}a^2$.

* This reduction is given to show that the integral is not infinite for the value $m = -1$, which is between the limits. See Art. 82.

Examples VIII.

1. Find the area included between the curve

$$a^2y = x^3 + ax^2,$$

and the axis of x. $\frac{a^2}{12}$.

2. Find the whole area of the curve

$$a^4y^2 = x^4(a^2 - x^2).$$ $\frac{\pi a^2}{4}$.

3. Find the area of a loop of the curve

$$x^2(a^2 + y^2) = y^2(a^2 - y^2).$$ $\frac{a^2}{2}(\pi - 2)$.

4. Find the area between the axes and the curve

$$y(x^2 + a^2) = b^2(a - x).$$ $b^2\left[\frac{\pi}{4} - \frac{\log 2}{2}\right]$.

5. Find the area between the curve

$$x^2y^2 + a^2y^2 - a^2x^2 = 0,$$

and one of its asymptotes. $2a^2$.

6. Find the area between the parabola $y^2 = 4ax$ and the straight line $y = x$. $\frac{8a^2}{3}$.

7. Find the area of the ellipse whose equation is

$$ax^2 + 2bxy + cy^2 = 1.$$ $\frac{\pi}{\sqrt{(ac - b^2)}}$.

8. Find the area of the loop of the curve

$$cy^2 = (x-a)(x-b)^2,$$

in which $c > 0$ and $b > a$. $\dfrac{8(b-a)^{\frac{5}{2}}}{15\sqrt{c}}$.

9. Find the area of the loop of the curve

$$a^3y^2 = x^4(b+x). \qquad \frac{32b^{\frac{1}{2}}}{105a^{\frac{3}{2}}}$$

10. Find the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} = 1. \qquad \frac{ab}{20}.$$

11. If n is an integer, prove that the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{n}} + \left(\frac{y}{b}\right)^{\frac{1}{n}} = 1$$

is

$$A = \frac{n(n-1)\cdots 1}{2n(2n-1)\cdots(n+1)}\,ab.$$

12. If n is an odd integer, prove that the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{n}} + \left(\frac{y}{b}\right)^{\frac{2}{n}} = 1$$

is

$$A = \frac{[n(n-2)\cdots 1]^2}{2n(2n-2)\cdots 2}\,\frac{\pi ab}{2}.$$

13. In the case of the curtate cycloid

$$x = a\psi - b\sin\psi, \qquad y = a - b\cos\psi,$$

find the area between the axis of x and the arc below this axis.

$$(2a^2 + b^2)\cos^{-1}\frac{a}{b} - 3a\sqrt{(b^2 - a^2)}.$$

14. If $b = \frac{1}{2}a\pi$, show that the area of the loop of the curtate cycloid is

$$\pi a^2\left[\frac{\pi^2}{8} - 1\right].$$

15. Find the area of the segment of the hyperbola

$$x = a\sec\psi, \qquad y = b\tan\psi,$$

cut off by the double ordinate whose length is $2b$.

$$ab\left[\sqrt{2} - \log\tan\frac{3\pi}{8}\right].$$

16. Find the whole area of the curve

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta. \qquad \frac{\pi}{2}(a^2 + b^2).$$

17. Find the area of a loop of the curve

$$r^2 = a^2\cos^2\theta - b^2\sin^2\theta. \qquad \frac{ab}{2} + \frac{(a^2 - b^2)}{2}\tan^{-1}\frac{a}{b}.$$

18. Find the areas of the large and of each of the small loops of the curve

$$r = a\cos\theta\cos 2\theta;$$

and show that the sum of the loops may be expressed by a single integral.

$$\frac{\pi a^2}{16} + \frac{a^2}{6}, \quad \text{and} \quad \frac{\pi a^2}{32} - \frac{a^2}{12}.$$

19. In the case of the spiral of Archimedes,

$$r = a\theta,$$

find the area generated by the radius vector of the first whorl and that generated by the difference between the radii vectores of the nth and $(n + 1)$th whorl.

$$\frac{4a^2\pi^3}{3} \quad \text{and} \quad 8na^2\pi^3.$$

20. Find the area of a loop of the curve

$$r = a \sin 3\theta. \qquad \frac{\pi a^2}{12}.$$

21. Find the area of the cardioid

$$r = 4a \sin^2 \tfrac{1}{2}\theta. \qquad 6\pi a^2.$$

22. Find the area of the loop of the curve

$$r = a \frac{\cos 2\theta}{\cos \theta}. \qquad \frac{a^2(4 - \pi)}{2}.$$

23. In the case of the hyperbolic spiral,

$$r\theta = a,$$

show that the area generated by the radius vector is proportional to the difference between its initial and its final value.

24. Find the area of a loop of the curve

$$r = a \cos n\theta.$$

$\frac{\pi a^2}{4n}$

25. Find the area of a loop of the curve

$$r^2 = a^2 \frac{\sin 3\theta}{\cos \theta}.$$

$\frac{a^2}{2}$

26. Find the area of a loop of the curve

$$r^2 \sin \theta = a^2 \cos 2\theta.$$

Notice that r *is real and finite from* $\theta = \frac{5\pi}{4}$ to $\theta = \frac{7\pi}{4}$, *and that* $\int \frac{d\theta}{\sin \theta}$ *is negative in this interval.*

$a^2\left[\sqrt{2} - \log(1 + \sqrt{2})\right]$.

27. Find the area of a loop of the curve

$$(x^2 + y^2)^2 = a^2 xy.$$

Transform to polar coordinates.

$\frac{a^2}{4}$

28. In the case of the limaçon

$$r = 2a \cos\theta + b,$$

find the whole area of the curve when $b > 2a$ and show that the same expression gives the sum of the loops when $b < 2a$.

$(2a^2 + b^2)\pi$.

29. Find separately the areas of the large and small loops of the limaçon when $b < 2a$.

If $\alpha = \cos^{-1}\left(-\frac{b}{2a}\right)$,

$$\text{large loop} = (2a^2 + b^2)\,\alpha + \frac{3b}{2}\sqrt{(4a^2 - b^2)};$$

$$\text{small loop} = (2a^2 + b^2)\,(\pi - \alpha) - \frac{3b}{2}\sqrt{(4a^2 - b^2)}.$$

30. Find the area of a loop of the curve

$$r^2 = a^2 \cos n\theta + b^2 \sin n\theta. \qquad \frac{\sqrt{(a^4 + b^4)}}{n}.$$

31. Find the area of the loop of the curve

$$r = a\frac{2\cos 2\theta - 1}{\cos\theta}, \qquad \left[5\sqrt{3} - \frac{8}{3}\pi\right]a^2.$$

32. Show that the sectorial area between the axis of x, the equilateral hyperbola

$$x^2 - y^2 = 1,$$

and the radius vector making the angle θ at the centre is represented by the formula

$$A = \frac{1}{4}\log\frac{1 + \tan\theta}{1 - \tan\theta};$$

and hence show that

$$x = \frac{e^{2A} + e^{-2A}}{2}, \qquad \text{and} \qquad y = \frac{e^{2A} - e^{-2A}}{2}.$$

If A *denotes the corresponding area in the case of the circle*

$$x^2 + y^2 = 1,$$

we have

$$x = \cos 2A, \qquad \text{and} \qquad y = \sin 2A.$$

In accordance with the analogy thus presented, the values of x *and* y *given above are called the hyperbolic cosine and the hyperbolic sine of* 2A. *Thus*

$$\frac{e^{2A} + e^{-2A}}{2} = \cosh(2A), \qquad \frac{e^{2A} - e^{-2A}}{2} = \sinh(2A).$$

33. Find the area of the loop of the curve

$$x^4 - 3axy^2 + 2ay^3 = 0. \qquad \frac{3^6 a^2}{35 \cdot 2^6}.$$

34. Find the area of the loop of the curve

$$x^{2n+1} + y^{2n+1} = (2n+1)\, ax^n y^n. \qquad \frac{2n+1}{2} a^2.$$

35. Find the area between the curve

$$x^{2n+1} + y^{2n+1} = (2n+1)\, ax^n y^n$$

and its asymptote. $\frac{2n+1}{2} a^2.$

36. Find the area of the loop of the curve

$$y^3 + ax^2 - axy = 0. \qquad \frac{a^2}{60}.$$

37. Find the area of a loop of the curve

$$x^4 + y^4 = a^2 xy. \qquad \frac{\pi a^2}{8}.$$

38. Trace the curve

$$x = 2a \sin\frac{y}{x},$$

and find the area of one loop. πa^2.

IX.

Volumes of Geometric Solids.

115. A geometric solid whose volume is required is frequently defined in such a way that the area of the plane section parallel to a fixed plane may be expressed in terms of the perpendicular distance of the section from the fixed plane. When this is the case, the solid is to be regarded as generated by the motion of the plane section, and its differential, when thus considered, is readily expressed.

116. For example, let us consider the solid whose surface is formed by the revolution of the curve APB, Fig. 17, about the axis OX. The plane section perpendicular to the axis OX is a circle; and if APB be referred to rectangular coordinates, the distance of the section from a parallel plane passing through the origin is x, while the radius of the circle is y. Supposing the centre of the section to move uniformly along the axis, the rate at which the volume is generated is not uniform, but its differential is the volume which would be generated while the centre is describing the distance dx, if the rate were made constant. This differential volume is therefore the cylinder whose altitude is dx, and the radius of whose base is y. Hence, if V denote the volume,

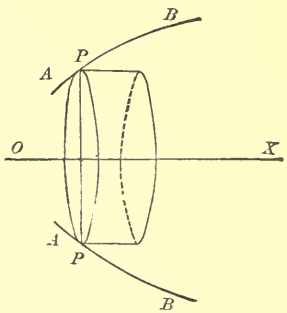

FIG. 17.

$$dV = \pi y^2 dx.$$

117. As an illustration, let it be required to find the volume of the paraboloid, whose height is h, and the radius of whose base is b.

The revolving curve is in this case a parabola, whose equation is of the form

$$y^2 = 4ax;$$

and since $y = b$ when $x = h$,

$$b^2 = 4ah, \qquad \text{whence} \qquad 4a = \frac{b^2}{h};$$

the equation of the parabola is therefore

$$y^2 = \frac{b^2}{h} x.$$

Hence the volume required is

$$V = \pi \int_0^h y^2 \, dx = \pi \frac{b^2}{h} \int_0^h x \, dx = \frac{\pi b^2 h}{2}.$$

118. It can obviously be shown, by the method used in Art. 116, that whatever be the shape of the section parallel to a fixed plane, *the differential of the volume is the product of the area of the generating section and the differential of its motion perpendicular to its plane.*

If the volume is completely enclosed by a surface whose equation is given in the rectangular coordinates x, y, z, and if we denote the areas of the sections perpendicular to the axes by A_x, A_y, and A_z, we may employ either of the formulæ

$$V = \int A_x \, dx, \qquad V = \int A_y \, dy, \qquad V = \int A_z \, dz.$$

The equation of the section perpendicular to the axis of x is determined by regarding x as constant in the equation of the surface, and its area A_x is of course a function of x.

For example, the equation of the surface of an ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The section perpendicular to the axis of x is the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - x^2}{a^2},$$

whose semi-axes are $\frac{b}{a}\sqrt{(a^2 - x^2)}$ and $\frac{c}{a}\sqrt{(a^2 - x^2)}$.

Since the area of an ellipse is the product of π and its semi-axes,

$$A_x = \frac{\pi bc}{a^2}(a^2 - x^2).$$

The limits for x are $\pm a$, the values between which x must lie to make the ellipse possible. Hence

$$V = \frac{\pi bc}{a^2}\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{4\pi abc}{3}.$$

119. The area A_x can frequently be determined by the conditions of the problem without finding the equation of the surface. For example, let it be required to find the volume of the solid generated by so moving an ellipse with constant major axis, that its center shall describe the major axis of a fixed ellipse, to whose plane it is perpendicular, while the extremities of its minor axis describe the fixed ellipse. Let the equation of the fixed ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let c be the major semi-axis of the moving ellipse. The minor semi-axis of this ellipse is y. Since the area of an ellipse is equal to π multiplied by the product of its semi-axes, we have

$$A_x = \pi cy = \frac{\pi cb}{a}\sqrt{(a^2 - x^2)}.$$

Therefore $$V = \frac{\pi bc}{a}\int_{-a}^{a}\sqrt{(a^2 - x^2)}\,dx;$$

hence, see formula (M),

$$V = \frac{\pi^2 abc}{2}.$$

The Solid of Revolution regarded as Generated by a Cylindrical Surface.

120. A solid of revolution may be generated in another manner, which is sometimes more convenient than the employment of a circular section, as in Art. 116. For example, let the cissoid POR, Fig. 18, whose equation is

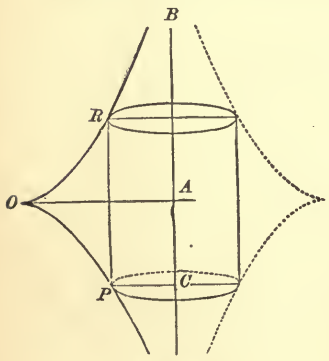

FIG. 18.

$$y^2(2a - x) = x^3,$$

revolve about its asymptote AB. The line PR, parallel to AB and terminated by the curve, describes a cylindrical surface. If we conceive the radius of this cylinder to pass from the value $OA = 2a$ to zero, the cylindrical surface will evidently generate the solid of revolution. Now every

point of this cylindrical surface moves with a rate equal to that of the radius; therefore the differential of the solid is the product of the cylindrical surface, and the differential of the radius. The radius and altitude in this case are

$$PC = 2a - x, \qquad \text{and} \qquad PR = 2y,$$

therefore $$V = 4\pi \int_0^{2a} (2ax - x^2)^{\frac{1}{2}} x \, dx.$$

Putting $$x - a = a \sin \theta,$$

$$V = 4\pi a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 \theta + \cos^2 \theta \sin \theta) \, d\theta = 2\pi^2 a^3.$$

Examples IX.

1. Find the volume of the spheroid produced by the revolution of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

about the axis of x. $\frac{4\pi ab^2}{3}$.

2. Find the volume of a right cone whose altitude is a, and the radius of whose base is b. $\frac{\pi ab^2}{3}$.

3. Find the volume of the solid produced by the revolution about the axis of x of the area between this axis, the cissoid

$$y^2 (2a - x) = x^3,$$

and the ordinate of the point (a, a). $8a^3\pi (\log 2 - \frac{2}{3})$.

4. Find the volume generated by the revolution of the witch,

$$y^2x - 2ay^2 + 4a^2x = 0,$$

about its asymptote.

See Art. 104. $4\pi^2a^3$.

5. The equilateral hyperbola

$$x^2 - y^2 = a^2$$

revolves about the axis of x: show that the volume cut off by a plane cutting the axis of x perpendicularly at a distance a from the vertex is equal to a sphere whose radius is a.

6. An anchor ring is formed by the revolution of a circle whose radius is b about a straight line in its plane at a distance a from its centre: find its volume. $2\pi^2ab^2$.

7. Express the volume of a segment of a sphere in terms of the altitude h and the radii a_1 and a_2 of the bases.

$$\frac{\pi h}{6}(h^2 + 3a_1^2 + 3a_2^2).$$

8. Find the volume generated by the revolution of the cycloid,

$$x = a(\psi - \sin\psi), \qquad y = a(1 - \cos\psi),$$

about its base. $5\pi^2a^3$.

9. The area included between the cycloid and tangents at the cusps and at the vertex revolves about the latter; find the volume generated.

π^2a^3.

10. Find the volume generated by the revolution of the part of the curve

$$y = e^x,$$

which is on the left of the origin, about the axis of x.

$\frac{\pi}{2}$.

11. The axes of two equal right circular cylinders, whose common radius is a, intersect at the angle α; find the volume common to the cylinders.

The section parallel to the axes is a rhombus. $\frac{16a^3}{3 \sin \alpha}$.

12. Find the volume generated by the revolution of one branch of the sinusoid,

$$y = b \sin \frac{x}{a},$$

about the axis of x. $\frac{\pi^2 a b^2}{2}$.

13. Find the volume enclosed by the surface generated by the revolution of an arc of a parabola about a chord, whose length is $2c$, perpendicular to the axis, and at a distance b from the vertex.

$\frac{16\pi b^2 c}{15}$.

14. Find the volume generated by the revolution of the tractrix, whose differential equation is

$$\frac{dy}{dx} = \pm \frac{y}{\sqrt{(a^2 - y^2)}},$$

about the axis of x.

Express $\pi y^2 dx$ in terms of y. $\frac{2\pi a^3}{3}$.

15. Find the volume generated by the curve

$$xy^2 = 4a^2 (2a - x)$$

revolving about its asymptote. $4\pi^2 a^3$.

16. Express the volume of a frustum of a cone in terms of its height h, and the radii a_1 and a_2 of its bases.

$\frac{\pi h}{3} (a_1^2 + a_1 a_2 + a_2^2)$.

17. Find the volume of a barrel whose height is $2h$, and diameter $2b$, the longitudinal section through the centre being a segment of an ellipse whose foci are in the ends of the barrel.

$$2\pi b^2 h \frac{2h^2+3b^2}{3(b^2+h^2)}.$$

18. Find the volume generated by the superior and by the inferior branch of the conchoid each revolving about the directrix; the equation, when the axis of y is the directrix, being

$$x^2y^2 = (a+x)^2(b^2-x^2).$$

$$\pi^2 ab^2 \pm \frac{4\pi b^3}{3}.$$

19. The area included between a quadrant of the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi,$$

and the tangents at its extremities revolves about the tangent at the extremity of the minor axis; find the volume generated.

$$\frac{\pi ab^2(10-3\pi)}{6}.$$

20. An ellipse revolves about the tangent at the extremity of its major axis; express the entire volume in the form of an integral, whose limits are 0 and 2π, and find its value. $2\pi^2 a^2 b$.

21. Find the volume generated by the revolution of a circular arc whose radius is a about its chord whose length is $2c$.

$$\frac{2\pi c(3a^2-c^2)}{3} - 2\pi a^2\sqrt{(a^2-c^2)}\sin^{-1}\frac{c}{a}.$$

22. A straight line of fixed length $2c$ moves with its extremities in two fixed perpendicular straight lines not in the same plane, and at a distance $2b$. Prove that every point in the moving line describes an ellipse in a plane parallel to both the fixed lines, and find the volume enclosed by the generated surface. $\frac{4\pi(c^2-b^2)b}{3}$.

X.

Double Integrals.

121. Let us consider the expression

$$\int_{y_1}^{y_2} \phi(x, y)\, dy, \quad \ldots\ldots\ldots \quad (1)$$

in which the limits y_1 and y_2 may be any functions of x. In the integration, x is to be treated like any other quantity independent of y which may be involved in the expression $\phi(x, y)$; in other words, as a constant with respect to the variable y. Thus the *indefinite* integral will contain both x and y; but since the *definite* integral is a function not of the independent variable but of the limits, the expression (1) is independent of y, but is generally a function of x. We may therefore denote it by $f(x)$, and write it in place of $f(x)$ in the expression of an integral in which x is the independent variable. Thus, putting

$$\int_{y_1}^{y_2} \phi(x, y)\, dy = f(x), \quad \ldots\ldots\ldots \quad (2)$$

$$\int_a^b f(x)\, dx = \int_a^b \int_{y_1}^{y_2} \phi(x, y)\, dy\, dx. \quad \ldots\ldots \quad (3)$$

In the last expression, which is called a *double integral*, x and y are two independent variables. It is to be noticed that the limits of the y-integration, which is to be performed first, may be functions of the other variable x, but the limits of the final integration must be constants, that is, independent both of x and of y.

122. In accordance with Art. 99, the expression (1) is the limit of $\Sigma_{y_1}^{y_2} \phi(x, y)\, \Delta y$, when Δy is diminished without limit, it being assumed that $\phi(x, y)$ is finite and continuous while y passes from y_1 to y_2. Further, assuming this to be the case

for all values of x from a to b (so that $f(x)$ in equation (3) is finite and continuous for all values concerned in the x-integration), the double integral is the limit of $\Sigma_a^b f(x)\,\Delta x$, when Δx is diminished without limit. It readily follows that the double integral is the limit of

$$\Sigma_a^b \Sigma_{y_1}^{y_2} \phi(x, y)\,\Delta y\,\Delta x,$$

where both Δy and Δx are diminished without limit.*

The typical term $\phi(x, y)\,\Delta y\,\Delta x$ is called *the element of the sum*, and in like manner $\phi(x, y)\,dy\,dx$ is *the element of the double integral.* As mentioned in Art. 100, in forming the expression for the element, no distinction need be drawn between any values of x and y, between x and $x + \Delta x$, y and $y + \Delta y$, respectively, because these distinctions disappear at the limit.

Limits of the Double Integral.

123. In discussing the limits of the double integral (3), Art. 121, it is convenient to consider first the simpler expression

$$\int_a^b \int_{y_1}^{y_2} dy\,dx. \quad . \quad . \quad . \quad . \quad (1)$$

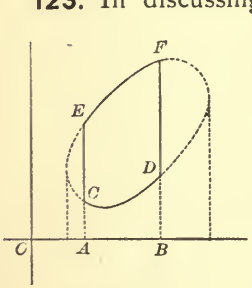

FIG. 19.

Performing the y-integration, this reduces to the simple integral

$$\int_a^b (y_2 - y_1)\,dx. \quad . \quad . \quad . \quad (2)$$

* Denoting the difference between $f(x)$ and $\Sigma_{y_1}^{y_2} \phi(x, y)\,\Delta y$, of which it is the limit, by e, e is a quantity which vanishes with Δy. Then the difference between $\Sigma_a^b f(x)\,\Delta x$ and $\Sigma_a^b \Sigma_{y_1}^{y_2} \phi(x, y)\,\Delta y\,\Delta x$ is $\Sigma_a^b e\,\Delta x$, a quantity which vanishes with Δy if a and b are finite. Hence, in this case, the double integral which is the limit of the first of these sums is also the limit of the second. But the conclusion does not follow when the limits are infinite; in fact, the double integral is not then always independent of the mode in which the limits become infinite.

In Fig. 19, using rectangular coordinates, let $OA = a$, $OB = b$, and let CD, EF be the curves $y = y_1$, $y = y_2$; then (2) is, by Art. 103, the expression for the area $CDFE$. Therefore the double integral (1) is represented by *the area enclosed by the curves* $y = y_1$, $y = y_2$ *and the straight lines* $x = a$, $x = b$.

124. In order that the double integral may represent the area enclosed by a single curve (like the dotted line of Fig. 19) of which the equation is known, the limits y_1 and y_2 must be two values of y corresponding to the same value of x, and the limits for x must be those values for which y_1 and y_2 are equal. Between the limiting values of x the values of y are real, and beyond them the values of y become imaginary. For example, suppose the curve to be the ellipse

$$2x^2 - 2xy + y^2 - 4x - 2y + 6 = 0;$$

solving for y,

$$y = x + 1 \pm \sqrt{(-x^2 + 6x - 5)} = x + 1 \pm \sqrt{[(x - 1)(5 - x)]};$$

whence

$$y_2 - y_1 = 2\sqrt{[(x - 1)(5 - x)]}.$$

This expression is real for all values of x between 1 and 5; hence the entire area is

$$A = \int_1^5 \int_{y_1}^{y_2} dy\, dx = 2\int_1^5 \sqrt{[(x - 1)(5 - x)]}\, dx^* = 4\pi.$$

It is evident that we might equally well have used the expression

$$A = \int_p^q \int_{x_1}^{x_2} dx\, dy = \int_p^q (x_2 - x_1) dy,$$

* It is useful to notice that a definite integral of this form, by a familiar property of the circle, represents the area of the semicircle whose diameter is the difference between the limits.

in which x_1 and x_2 are obtained as functions of y, by solving the equation of the curve for x, thus

$$x = \tfrac{1}{2}(y + 2) \pm \tfrac{1}{2}\sqrt{(-y + 8y - 8)};$$

and the limiting values of y are those obtained by putting the radical equal to zero, namely, $p = 4 - 2\sqrt{2}$, $q = 4 + 2\sqrt{2}$. Accordingly the same area is expressed by

$$A = \int_{4-2\sqrt{2}}^{4+2\sqrt{2}} \sqrt{(-y^2 + 8y - 8)},$$

whence $A = 4\pi$ as before.

125. It appears therefore that no distinction is to be drawn between the expressions

$$\iint dy\,dx \quad \text{and} \quad \iint dx\,dy.$$

We may in fact regard either one as representing any area whatever, the value becoming definite only when we assign a defined closed *contour* or boundary line; just as $\int dx$ may represent a line of any length measured along the axis of x, and is only definite when we assign the limits which determine its two extremities. Thus the contour bears the same relation to the double integral that the pair of limits does to the simple integral.

When the boundary of a given area is made up of lines having different equations, it is not generally possible to express the area by means of a single double integral. This was possible, it is true, in the case of the area enclosed by the full line in Fig. 19, because the equations of two of the bounding lines, $x = a$ and $x = b$, contained only one variable, and the integration with respect to this was made the final one in expression (1). But, if integration with respect to x had been performed first, it would have been necessary to break the area

up into several parts, of which it is the algebraic sum. In the expressions for these parts, the limits for x would be taken from the equations of the different lines, and the final limits would be the ordinates, which it would be necessary to find (instead of the known abscissas), of the intersections C, D, E, and F.

126. Returning now to the general double integral

$$\int_a^b \int_{y_1}^{y_2} \phi(x, y) dy\, dx, \quad \ldots \ldots \quad (1)$$

and employing rectangular coordinates, let the area $ASBR$ in the plane of xy, Fig. 20, be that which is defined by the limits of the given integral, in other words, the area represented by

$$\int_a^b \int_{y_1}^{y_2} dy\, dx. \quad \ldots \quad (2)$$

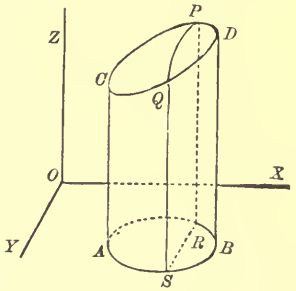

FIG. 20.

In evaluating the double integral (1) with the given limits, the integration of the element $\phi(x, y)dy\, dx$ is said *to extend over the area* $ASBR$ represented by expression (2).

Now let us construct the surface whose equation is

$$z = \phi(x, y). \quad \ldots \ldots \quad (3)$$

The z-coordinates of all points whose projections on the plane of xy are on the curve $ASBR$ lie in a cylindrical surface, which in Fig. 20 is supposed to cut the surface (3) in the curve $CQDP$. It is assumed that $\phi(x, y)$, or z, remains finite and continuous while the independent variables x and y vary in any way, provided the point (x, y) remains within the area $ASBR$. This assumption is clearly identical with that made in Art. 122. A definite solid will then be enclosed between the surface (3),

the plane of xy, and the cylindrical surface. We have now to show that this solid will represent the double integral (1).

Let $SRPQ$ be a section of this solid by a plane parallel to that of yz, so that in it x has a constant value. The ordinates of the points R and S are the values of y_1 and y_2 corresponding to that particular value of x. The section $SRPQ$ may be regarded as generated by the line z, while y varies from y_1 to y_2; hence, denoting its area by A_x, as in Art. 118,

$$A_x = \int_{y_1}^{y_2} z\, dy.$$

Therefore the volume is

$$V = \int_a^b A_x dx = \int_a^b \int_{y_1}^{y_2} z\, dy\, dx,$$

which is identical with expression (1).

Change of the Order of Integration.

127. It is obvious that the order of integration may be reversed as in Art. 124, provided that the integration extends over the same area. Considering the corresponding process of double summation, we may be said, in the first case, to sum up those of the elements $z\, dy\, dx$ which have a common value of x into the sum $A_x dx$, which then constitutes the element of the second summation; while in the second case we first sum up those of the original elements which have a common value of y into the element $A_y dy$, which is afterward summed for all values of y within the given limits.

Since, in expression (1), Art. 126, dx is a constant with respect to the y-integration, it may be removed to the left of the corresponding integral sign. In the resulting expression,

$$\int_a^b dx \int_{y_1}^{y_2} \phi(x, y) dy,$$

it is to be understood that because the y-integral which is a function of x *follows* the sign of x-integration it is "under" it, or subject to it. This notation has the advantage of making it perfectly clear to which variable each integral sign corresponds.*

128. The limits of a double integral are sometimes given in the form of a restriction upon the values which the independent variables x and y can simultaneously assume. Suppose, for example, that in the integration neither variable can become negative or exceed a, and that y cannot exceed x. If x and y are coordinates of a point in Fig. 21, the restriction is equivalent to saying that the point cannot be below the axis of x, to the right of the line $x = a$ or above the line $y = x$. Therefore the area of integration is the triangle OAB. This may be expressed either by giving to y the limits zero and x, and then giving to x the limits zero and a; or, reversing the order of integration, by giving to x the limits y and a, and then giving to y the limits zero and a. Thus the area of integration can be represented by either of the expressions

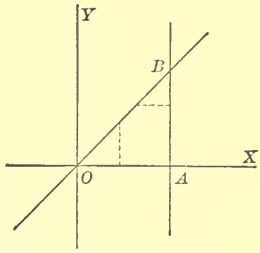

FIG. 21.

$$\int_0^a \int_0^x dy\, dx, \quad \text{or} \quad \int_0^a \int_y^a dx\, dy.$$

129. As an illustration, let us take the integral

$$U = \int_0^1 \int_0^x \sin^{-1} y \sqrt{(1 - x)} \sqrt{(x - y)} dy\, dx.$$

* We have in the preceding articles supposed the first written integral sign to correspond to the last written differential when the pair of differentials is written last; but writers are not uniform in this respect.

The y-integration, which is here indicated as the first to be performed, namely, the integration of $\sqrt{(x-y)}\sin^{-1}y\,dy$ is impracticable. Noticing, however, that the x-integration in the given expression is of a known or integrable form, we change the order of integration, determining the new limits so as to represent the same area of integration. Since this area is that indicated in Fig. 21, when $a = 1$, we thus obtain

$$U = \int_0^1 \sin^{-1}y\,dy \int_y^1 \sqrt{[(1-x)(x-y)]}\,dx.$$

The value of the x-integral is $\frac{1}{8}\pi(1-y)^2$; hence

$$U = \frac{\pi}{8}\int_0^1 \sin^{-1}y(1-y)^2 dy.$$

Finally, integrating by parts and then putting $y = \sin\theta$,

$$U = \frac{\pi}{24}\int_0^1 \frac{(1-y)^3 dy}{\sqrt{(1-y^2)}} = \frac{\pi}{24}\int_0^{\frac{\pi}{2}} (1-\sin\theta)^3 d\theta = \frac{\pi}{24}\left[\frac{5\pi}{4} - \frac{11}{3}\right].$$

130. In a case where the first integration can be effected with respect to either variable, it may happen that in one case only are the limits such as to make the second integration possible. Given, for example,

$$U = \int_0^\infty \int_\alpha^\beta e^{-xy}\,dy\,dx, \quad \ldots\ldots \quad (1)$$

in which α and β are both positive. Integration for y gives

$$U = \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx, \quad \ldots\ldots \quad (2)$$

and for this form the second integration is impracticable. But, integrating expression (1) first with respect to x, we have,

owing to the special values (zero and infinity) of the limits, the simple form

$$U = \int_{\alpha}^{\beta} \frac{dy}{y} = \log \frac{\beta}{\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The double integration in this example extends over the infinite strip of area between the parallel lines $y = \beta$, $y = \alpha$ and on the right of the axis of y, yet we have obtained a finite result. It is plain that this would not have been possible but for the fact that the element $e^{-xy}\,dy\,dx$ approaches to the limit zero, when x increases without limit (y remaining between finite limits); that is, the element vanishes for all the infinitely distant points which are included in the area of integration.

The fact that we have thus obtained the value of the definite simple integral (2), although the corresponding indefinite integral could not be found, will be noticed in connection with the methods of evaluating definite integrals.

Triple Integrals.

131. An expression of the general form

$$\int_{a}^{b}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \phi(x, y, z)\, dz\, dy\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is called a *triple integral.* In the first integration the limits z_1 and z_2 are in general functions of x and y. In the next, the limits y_1 and y_2 are in general functions of x; and in the final integration the limits are constants.

It is readily seen that the triple integral, like the double integral considered in Art. 122, is the limit (where Δx, Δy, and Δz diminish without limit) of the result of a corresponding summation. Accordingly $\phi\,(x, y, z)\,dz\,dy\,dx$ is called *the element* of the integral (1).

Consider now the triple integral with the same limits, but having the simpler element $dz\,dy\,dx$; the z-integration can here be effected at once, and we have

$$\int_a^b\int_{y_1}^{y_2}\int_{z_1}^{z_2} dz\,dy\,dx = \int_a^b\int_{y_1}^{y_2}(z_2 - z_1)\,dy\,dx. \quad . \quad . \quad . \quad (2)$$

Now supposing x, y, and z to be rectangular coordinates in space, the last expression represents the difference between two solids defined as in Art. 126; that is, the triple integral in equation (2) represents *the volume included between the surfaces* $z = z_1$, $z = z_2$, *and the cylindrical surface whose section in the plane of* xy *is the contour determined by the limits for* y *and* x.

132. When the volume is completely enclosed by a surface defined by a single equation or relation between x, y, and z, the case is analogous to that of the area discussed in Art. 124; z_1 and z_2 will be two values of z corresponding to the same values of x and y, and the limits for x and y must be determined by the area within which the value of $z_2 - z_1$ is real. This area is the section with the plane of xy of a cylinder which circumscribes the given volume.

In this case it, is plain that we may perform the integrations in any order, provided we properly determine the limits. Considering the corresponding process of summation, we may be said in equation (2) to have summed those of the *ultimate elements* which have common values of x and y into the *linear element* $(z_2 - z_1)\,dy\,dx$. If the integration for y is effected next, we combine such of these last elements as have a common value of x into the *areal element* $A_x\,dx$ which is often called a *lamina*. Thus each of the formulæ in Art. 118 is the result of performing two of the integrations in the general expression for a volume, namely,

$$V = \iiint dx\,dy\,dz.$$

In any case where the result of two integrations (that is, the area of a section) is known, we may of course take advantage of this and pass at once to the simple integral, as in Art. 118, where in finding the volume of the ellipsoid we regarded the expression for the area of an ellipse as known.

Integration over a Known Volume.

133. In the general expression, (1) of Art. 131, the integration is said *to extend over the volume defined by the limits*, that is, the volume represented by either member of equation (2). It is sometimes possible, in the case of a triple integral of the more general form, to take advantage of our knowledge of the geometrical solid over which the integration extends. For example, let it be required to evaluate the integral

$$U = \int\int\int x^2 dx\, dy\, dz$$

for all values of x, y, and z such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

does not exceed unity, that is to say, when the integration extends over the volume of the ellipsoid considered in Art. 118. We here perform the integration for y and z first, because we see that the result will be the section A_x of this solid. This reduces the integral to the simple form

$$U = \int x^2 A_x\, dx = \frac{\pi bc}{a^2} \int_{-a}^{a} x^2(a^2 - x^2) dx$$

$$= \pi a^3 bc\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{4\pi a^3 bc}{15}.$$

134. A transformation of variables will sometimes enable us to make use of geometrical considerations. For example, let us find the value of

$$U = \iiint \frac{dx\,dy\,dz}{\sqrt{(xyz)}}$$

for all positive values of the variables whose sum does not exceed a; or, as it may be written,

$$U = \int_0^a \frac{dx}{\sqrt{x}} \int_0^{a-x} \frac{dy}{\sqrt{y}} \int_0^{a-x-y} \frac{dz}{\sqrt{z}}.$$

If we put $\sqrt{x} = \xi$, $\sqrt{y} = \eta$, $\sqrt{z} = \zeta$, we have

$$U = 8 \int_0 \int_0 \int_0 d\xi\, d\eta\, d\zeta,$$

and the condition determining the upper limits is

$$\xi^2 + \eta^2 + \zeta^2 = a.$$

The last integral (regarding ξ, η, and ζ as a new set of rectangular coordinates) represents one octant of a sphere whose radius is $\sqrt{a}$; hence

$$U = \tfrac{4}{3}\pi a^{\frac{3}{2}}.$$

Representation of a Triple Integral by a Mass of Variable Density.

135. We may, in any triple integral of the general form

$$\iiint \phi(x, y, z)\, dx\, dy\, dz,$$

suppose the function $\phi(x, y, z)$ to represent the density of the

geometric element $dx\,dy\,dz$ at the point (x, y, z),* so that the element of the integral is the element of *a mass of variable density* occupying the space defined by the limits of the integral. This mass will therefore represent the value of the triple integral.

For example, the integral U of Art. 134 extends over the volume of integration enclosed by the three coordinate planes and the plane

$$x + y + z = a;$$

and, since $\phi(x, y, z) = (xyz)^{-\frac{1}{2}}$, the indefinite integral represents a mass whose density varies inversely as the square root of the product of the coordinates, and is unity at the point (1, 1, 1). Therefore the definite integral U represents the mass of the tetrahedron or triangular pyramid bounded as above and having this law of variable density.

It may be noticed that, in the transformation employed in evaluating U, the solid was converted without change of mass into an octant of a sphere having the uniform density 8.

136. When the variable corresponding to the integration indicated as the final one is not involved in the limits of either of the other variables, none of the limits will be changed if we effect the integration with respect to this variable first. For example, given the integral

$$U = \int_0^h dx \int_0^a dy \int_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (x^2 + y^2 + z^2)dz. \quad . \quad . \quad (1)$$

Because x does not occur in the limits for y and z, the yz-integration extends over the same area, no matter what the

* In the element of the sum of which the triple integral is the limit, $\phi(x, y, z)$ is to be taken as the *average density* of the element $\Delta x\,\Delta y\,\Delta z$, so that $\phi(x, y, z)\,\Delta x\,\Delta y\,\Delta z$ shall be the mass of the geometric element, that is to say, *the element of mass*. But this average density is the density at some point *within* the element $\Delta x\,\Delta y\,\Delta z$, and the distinction between all such points disappears at the limit as in the case of the simple integral. See Art. 100.

value of x; namely, in this case, a semicircle whose radius is a. This is as much as to say that the volume of integration is bounded by a cylindrical surface, whose elements are parallel to the axis of x, together with planes parallel to the plane of yz.* (In the example, it is the half of a circular cylinder of height h.) Hence we may write

$$U = \int_0^a dy \int_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} dz \int_0^h (x^2+y^2+z^2)dx$$

$$= \frac{h^3}{3}\int_0^a \int_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} dz\,dy + h\int_0^a \int_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (y^2+z^2)dz\,dy. \quad (2)$$

The first of these double integrals is the area of the semicircle mentioned above Hence

$$U = \frac{\pi h^3 a^2}{6} + 2h\int_0^a y^2\sqrt{(a^2-y^2)}dy + \frac{2h}{3}\int_0^a (a^2-y^2)^{\frac{3}{2}}dy,$$

and finally, $U = \frac{1}{12}\pi h a^2(3a^2+2h^2)$.

Examples X.

1. Find the volume cut from a right circular cylinder whose radius is a, by a plane passing through the centre of the base and making the angle α with the plane of the base. $\frac{2}{3}a^3 \tan\alpha$.

2. Show that the integral of $x\,dx\,dy$ over any area symmetrical to the axis of y vanishes. Interpret the result geometrically, and apply to find the integral of $(c+mx+ny)dx\,dy$ over the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2. \qquad \pi abc.$$

* The case is analogous to that of the double integral when both pairs of limits are constants, that is, when the area of integration is a rectangle.

3. Show that the volume between the surface

$$z^n = a^2x^2 + b^2y^2$$

and any plane parallel to the plane of xy is equal to the circumscribing cylinder divided by $n + 1$.

4. Find the volume enclosed by the surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1. \qquad \frac{8\pi abc}{5}.$$

5. A moving straight line, which is always perpendicular to a fixed straight line through which it passes, passes also through the circumference of a circle whose radius is a, in a plane parallel to the fixed straight line and at a distance b from it; find the volume enclosed by the surface generated and the circle. $\frac{\pi a^2 b}{2}$.

6. A cylinder cuts the plane of xy in an ellipse whose semi-axes are a and b, and the plane of xz in an ellipse whose semi-axes are a and c, the elements of the cylinder being parallel to the plane of yz; find the volume of the portion bounded by the semi-ellipses and the surface. $\frac{2}{3}abc$.

7. Find the volume enclosed by the surface

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$$

and the plane $x = a$. $\frac{\pi abc}{2}$.

8. Find the volume enclosed by the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Find A_z as in Art. 107. $\frac{4\pi a^3}{35}$.

9. Find the volume between the coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1. \qquad \frac{abc}{90}.$$

10. Find the volume cut from the paraboloid of revolution

$$y^2 + z^2 = 4ax$$

by the right circular cylinder

$$x^2 + y^2 = 2ax,$$

whose axis intersects the axis of the paraboloid perpendicularly at the focus, and whose surface passes through the vertex.

$$2\pi a^3 + \frac{16a^3}{3}.$$

11. Find the volume cut from a sphere whose radius is a by a right circular cylinder whose radius is b, and whose axis passes through the centre of the sphere.

$$\frac{4\pi}{3}\left[a^3 - (a^2 - b^2)^{\frac{3}{2}}\right].$$

12. Prove that the volume generated by the revolution about the axis of y of the area between the equilateral hyperbola

$$x^2 - y^2 = a^2$$

and the double ordinate $2y_1$ is equal to the sphere of radius y_1. [Ex. 11 shows that the circle has the same property.]

13. Change the order of integration in the double integral

$$\int_0^a \int_y^{2a-y} \phi(x, y)\,dx\,dy.$$

$$\int_0^a \int_0^x \phi(x, y)\,dy\,dx + \int_a^{2a} \int_0^{2a-x} \phi(x, y)\,dy\,dx.$$

14. Evaluate the integral

$$\int_0^a \int_y^{2a-y} \frac{dx\,dy}{\sqrt{(x^2 - xy)}}. \qquad 2\sqrt{2}\,\log(1 + \sqrt{2})a.$$

15. Integrate $xy\,dy\,dx$ over the area of the circle

$$(x - h)^2 + (y - k)^2 = a^2. \qquad \pi a^2 hk.$$

16. Show that the integral of the element in Ex. 15 over *any* square circumscribing the given circle is $4a^2hk$.

17. Integrate $x^3y\,dy\,dx$ over the circle of Ex. 15, and over the circumscribing square with sides parallel to the axes.

$$\pi a^2hk(h^2 + \tfrac{5}{4}a^2);\quad 4a^2hk(h^2 + a^2).$$

18. Evaluate $\iiint xyz\,dx\,dy\,dz$ for all positive values of x, y, and z whose sum is less than a; also for all positive values the sum of whose squares is less than a^2. $\frac{a^6}{720};\ \frac{a^6}{48}$.

19. Evaluate $\iiint x^2yz\,dx\,dy\,dz$ for all positive values subject to the condition

$$x^2 + y^2 + z^2 < a^2. \qquad \frac{a^7}{105}.$$

20. Evaluate $\iiint \sqrt{\frac{x}{yz}}\,dx\,dy\,dz$ for all positive values subject to the condition

$$x + y + z < 1. \qquad \tfrac{1}{2}\pi.$$

21. Find the volume of a cavity just large enough to permit of the complete revolution of a circular disk of radius a, whose centre describes a circle of the same radius, while the plane of the disk is constantly parallel to a fixed plane perpendicular to that of the circle in which its centre moves. $\frac{2}{3}a^3(3\pi + 8)$.

22. An inverted hollow circular cone of radius a and height h is filled with material of which the density varies as the depth below the plane of the base; find the mass of the material, μ being the density at the vertex. $\frac{1}{12}\pi\mu a^2h$.

23. A vessel consisting of a hemisphere of radius a and a cylinder having the same base and height h is filled with material having the same law of density as in Ex. 22; what is the value of h if half the mass is in the hemisphere? $(\frac{2}{3} + \frac{1}{6}\sqrt{34})a$.

XI.

The Polar Element of Area.

137. When polar coordinates are used, if concentric circles be drawn, corresponding to values of the radius vector r having the common difference Δr, and then straight lines through the pole corresponding to values of the angular coordinate θ having the uniform difference $\Delta\theta$, any given portion of the plane may be divided, as in Fig. 22, into small areas. The value of any one of these is readily seen to be $\Delta r \cdot r\Delta\theta$,

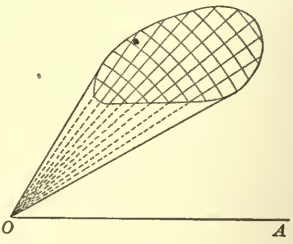

FIG. 22.

where r has a value midway between the greatest and least values of r in the area in question. It follows that the result

$$\Sigma\Sigma r\Delta r\Delta\theta$$

of a double summation of this element between proper limits will give an approximate expression for the given area. Hence the double integral

$$A = \iint r\,dr\,d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which (compare Art. 122) is the limit of this expression when Δr and $\Delta\theta$ are both diminished without limit, is the exact expression for the given area.

138. The differential expression $r\,dr\,d\theta$, or *polar element of area*, is the product of the differentials dr and $r\,d\theta$, which correspond to the mutually rectangular dimensions Δr and $r\Delta\theta$ of the element of summation. These factors are the differentials of the motions of the point (r, θ), respectively, when r alone

varies and when θ alone varies. It is obvious that the element of area for any system of coordinates can in like manner be shown to be the product of the corresponding differential motions, provided only that these motions are *at right angles to one another.** The element is in such a case said to be *ultimately* a rectangle.

139. The formula used in section VIII is in fact the result of performing the integration for r in formula (1) above. Thus when the pole is outside of the given area, as in Fig. 22, we have

$$A = \int\int_{r_1}^{r_2} r\,dr\,d\theta = \tfrac{1}{2}\int (r_2^2 - r_1^2)d\theta,$$

the formula of Art. 110. The limits for θ are now the values which make $r_2 = r_1$; that is, in the case of a continuous curve, the values for which the radius vector is tangent to the curve. So also when, as in the example of Art. 109, the pole is on the curve (so that $r_1 = 0$ for all values of θ), the limits for θ correspond to tangents to the curve. But, when the pole is within the curve, we assume the lower limit zero to avoid negative values of r, and then make θ vary through a complete revolution, that is, from 0 to 2π.

140. We may, of course, in formula (1) integrate first for θ, the limits being functions of r. Thus

$$A = \int\int_{\theta_1}^{\theta_2} r\,d\theta\,dr = \int (\theta_2 - \theta_1) r\,dr.$$

This corresponds to summation of the elements along a circular arc of radius r, so as to form the element $(\theta_2 - \theta_1)r\,dr$, in which the angular limits are two consecutive values of θ corre-

* In other words, whenever the loci of constant values of one coordinate cut orthogonally the like loci for the other coordinate; as in the case of the coordinates latitude and longitude on a spherical surface, or on a map made on any system in which the representatives of the parallels and meridians cut at right angles.

sponding to the same value of r, and such that the arc lies within the given area. If the pole is outside of the curve, the limits of the final integration will be the greatest and least values of r, for each of which $\theta_2 - \theta_1$ will vanish; but if the pole is within the curve, the arc will at the lower limit become a complete circumference, and the integral will represent the area included between the given curve and this circumference.

Transformation of a Double Integral.

141. We have seen in Art. 126 that a double integral of the general form

$$\iint \phi(x, y)dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may, by taking x and y as rectangular coordinates, be represented by the volume of a cylindrical solid whose base is the area in the plane of xy determined by the limits of integration, and whose upper surface is defined by the equation $z = \phi(x, y)$. If we introduce the new independent variables r and θ connected with x and y by the usual polar formulæ,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

it is obvious that the same solid will represent the integral

$$\iint \phi(r\cos\theta, r\sin\theta)r\,dr\,d\theta, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(in which the element of area $dx\,dy$ is replaced by the polar element $r\,dr\,d\theta$), provided the integration extends over the same area, of which the boundary must of course now be expressed in polar coordinates.

This transformation is often useful when the equation of the boundary is simplified by the use of polar coordinates, par-

ticularly when ϕ is also simplified. For example, the last of the integrals in equation (2), Art. 136, may be transformed to

$$h\int_0^{\pi}\int_0^{a} r^2 . r\,dr\,d\theta = \frac{\pi h a^4}{4},$$

where the limits correspond to the semicircle whose radius is a.

142. The polar element of area, which we have obtained geometrically in Art. 137, may also be derived by transformation from $dx\,dy$, by the method given below, which is applicable to any transformation of variables.

Since the element depends upon the independent variation of two quantities, it is necessary to replace the differentials, one at a time, by those of the new variables; the other variable at each stage being regarded as constant. Let us first replace dy by $d\theta$. Expressing y in terms of x and θ, by the equations of transformation, we have

$$y = x \tan \theta;$$

and, differentiating this, x being regarded as constant,

$$dy = x \sec^2 \theta\, d\theta.$$

The element $dx\,dy$ now takes the form

$$dx\, x \sec^2 \theta\, d\theta.$$

Again, differentiating

$$x = r \cos \theta,$$

θ being now regarded as constant, we have

$$dx = \cos \theta\, dr,$$

and, substituting the value of $x\,dx$, the element becomes $r\,dr\,d\theta$.*

* If we first replace dx by dr, and then dy by $d\theta$, we shall obtain different expressions for the differentials, but the same product. But, if we replace dy by dr, and then dx by $d\theta$, the result will be $-r\,dr\,d\theta$. The negative sign is due to the fact that the mode of measuring θ is such as to make it decrease with the increase of x, when y is positive.

Cylindrical Coordinates.

143. In determining the volume of a solid, it is sometimes convenient to use the polar coordinates r and θ in place of x and y, while retaining the third coordinate z perpendicular to the plane of $r\theta$. The system r, θ, z is sometimes called that of *cylindrical coordinates*, because the locus of $r =$ a constant is the surface of a right circular cylinder. The element of volume in this system is obviously the product of dz and the polar element of area, that is,

$$r\,dr\,d\theta\,dz,$$

which has the ultimate form of a rectangular parallelopiped.

When this element is used in finding the volume of a solid bounded in part by a cylindrical surface of which the elements are parallel to the axis of z, like that represented in Fig. 20, Art. 126, the integration for z is naturally performed first, its limits being the values of z in terms of r and θ given by the equations of the surfaces which cut the cylindrical surface.

144. For example, let us find the volume cut from a sphere of radius a by a right circular cylinder having a radius of the sphere for one of its diameters. Taking the centre of the sphere as origin, the diameter of the cylinder as initial line, and the axis of z parallel to the axis of the cylinder, we have, for the equation of the sphere, in these coordinates

$$z^2 + r^2 = a^2, \quad \ldots\ldots\ldots \quad (1)$$

and, for that of the cylinder,

$$r = a\cos\theta. \quad \ldots\ldots\ldots \quad (2)$$

The limits for z are now taken from equation (1), those of r are zero and that given by equation (2), and those of θ are $\pm \frac{1}{2}\pi$

which make r in equation (2) vanish. Hence we have for the required volume

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} r\,dr \int_{-\sqrt{(a^2-r^2)}}^{\sqrt{(a^2-r^2)}} dz$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} (a^2 - r^2)^{\frac{1}{2}} r\,dr$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[-\tfrac{1}{3}(a^2 - r^2)^{\frac{3}{2}} \right]_0^{a\cos\theta}.$$

From the symmetry of the solid it is apparent that we may take the limits for θ to be 0 and $\frac{1}{2}\pi$, and double the result.* Thus

$$V = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} (1 - \sin^3\theta)d\theta = \frac{2\pi a^3}{3} - \frac{8a^3}{9}.$$

145. In the system of cylindrical coordinates z and r may be regarded as rectangular coordinates in a plane which passes through the axis of z, and makes the diedral angle θ with a fixed plane. An equation between z and r independent of θ is thus the equation of *a surface of revolution* about the axis of z, and at the same time it is, in rectangular coordinates, the equation of the generating curve (positive values of r only being admissible). The volume of the solid of revolution is therefore

* In this example, the result, which would otherwise have been written in the form $V = \frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^3\theta)d\theta$, would not have been correct, because the $(a^2 - r^2)^{\frac{3}{2}}$ which occurs in the integration stands for the *positive* value of the radical, whereas in the fourth quadrant $a \sin\theta$, which we have put for it, stands for the *negative* value.

given by the present form of triple integral when the limits of θ are 0 and 2π, and the rz-integration extends over the area of the generating curve. Thus

$$V = \int_0^{2\pi} d\theta \iint r\,dr\,dz = 2\pi \iint r\,dr\,dz,$$

since by hypothesis θ is not involved in the limits of either of the other variables (compare Art. 136).

In the last written expression for V, if zero is the lower limit of the integration for r, the element of final integration is $\pi r^2 dz$, equivalent to the lamina of Art. 116; but, if we integrate first for z, we have for the final integration the cylindrical element $2\pi(z_2 - z_1)r\,dr$, corresponding to the method employed in Art. 120.

Solids of Revolution with Polar Coordinates.

146. The volume of the solid produced by the revolution about the initial line of a curve given in polar coordinates may be expressed by a double integral of which the element is derived from the polar element of area. In this revolution, every point P of the polar element of summation, $r\,\Delta r\,\Delta\theta$, constructed in Fig. 23, describes a circle whose radius is $PR = r \sin\theta$, and moves always *in a direction perpendicular to the plane of the element.* Therefore the volume described by the element is equal to its area multiplied by $2\pi r \sin\theta$, where r and θ belong to some point within the element. Since the distinction between all such points disappears at the limit, we have for the element of volume

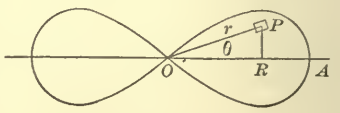

FIG. 23.

$$2\pi r^2 \sin\theta\,dr\,d\theta;$$

and the integral of this expression over the given area is the required volume.

147. For example, the curve in Fig. 23 being the lemniscate

$$r_1^2 = a^2 \cos 2\theta,$$

the volume generated by the right-hand loop, or rather by the half-loop in the first quadrant, is

$$V = 2\pi \int_0^{\frac{\pi}{4}} \int_0^{r_1} r^2 \sin\theta \, dr \, d\theta = \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \sin\theta \, d\theta.$$

Transforming, by putting $x = \cos\theta$, since

$$\cos 2\theta = 2\cos^2\theta - 1,$$

$$V = \frac{2\pi a^3}{3} \int_{\sqrt{\frac{1}{2}}}^{1} (2x^2 - 1)^{\frac{3}{2}} dx;$$

and again, by putting $x\sqrt{2} = \sec\phi$,

$$V = \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \tan^4\phi \sec\phi \, d\phi = \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \frac{\sin^4\phi}{\cos^5\phi} d\phi.$$

Using now the formula of reduction (6), Art. 70, we find

$$\int \frac{\sin^4\phi}{\cos^5\phi} d\phi = \frac{\sin^3\phi}{4\cos^4\phi} - \frac{3}{8}\frac{\sin\phi}{\cos^2\phi} + \frac{3}{8}\log\frac{1+\sin\phi}{\cos\phi};$$

whence, applying the limits,

$$V = \frac{\pi a^3 \sqrt{2}}{3}\left[\frac{3}{8}\log_e(\sqrt{2}+1) - \frac{\sqrt{2}}{8}\right] = 0.228 \times a^3.$$

Polar Coordinates in Space.

148. The double integral employed in the preceding article is in fact an application of *the polar system of coordinates in space.* In this system, a point is determined by polar coordinates in a plane which passes through a fixed axis and makes the diedral angle ϕ with a fixed plane of reference passing through the axis. In Art. 146, the fixed axis is the initial line

OA, and taking the plane of the paper as the plane of reference, the third coordinate ϕ is the angle described by the upper half-plane in the revolution of the figure. The differential of the motion of P when ϕ varies is $r \sin\theta\, d\phi$, and this motion is at right angles to the plane of the differentials dr and $r\, d\theta$; hence the ultimate element of volume is $r^2 \sin\theta\, dr\, d\theta\, d\phi$. But in the case of the solid of revolution the limits of the ϕ-integration are 0 and 2π, so that the result of this integration is $2\pi r^2 \sin\theta\, dr\, d\theta$, the element of the double integral derived in Art. 146.

149. In the equations of transformation connecting these coordinates with the rectangular system, it is usual to take the axis of z as that about which the angle ϕ is described, and the plane xz as that for which $\phi = 0$. Then in the plane ZOR, Fig. 24, the line OR is usually taken as the initial line for the polar coordinates ρ and θ, so that the three coordinates of P are

$$\rho = OP, \quad \theta = POR, \quad \phi = AOR.$$

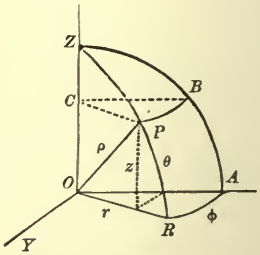

FIG. 24.

Thus ρ in this system is the distance of the point P from a fixed pole, and θ and ϕ correspond to the spherical coordinates latitude and longitude on a sphere whose radius is ρ. Denote the radius of the circle of latitude BP by r; then

$$r = \rho \cos\theta,$$

and r and ϕ are the polar coordinates of the projection of P in the plane of xy. Hence the formulæ connecting the rectangular coordinates x, y, z with ρ, θ and ϕ are

$$x = r\cos\phi = \rho\cos\theta\cos\phi,$$

$$y = r\sin\phi = \rho\cos\theta\sin\phi,$$

$$z = \rho\sin\theta.$$

The differentials of the motions of P when one of the coordinates ρ, θ, and ϕ varies, the other two remaining constant, are, respectively,

$$d\rho, \qquad \rho\, d\theta, \qquad \text{and} \qquad \rho \cos\theta\, d\phi;$$

hence the element of volume is their product,

$$\rho^2 \cos\theta\, d\rho\, d\theta\, d\phi,$$

in which the factor $\cos\theta$ occurs, in place of the factor $\sin\theta$ which appears in the element derived in the preceding article, because the axis of rotation is now perpendicular to the initial line.

In the case of a sphere whose centre is at the pole, all the limits are constant, and we have

$$V = \int_0^a \rho^2 d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^3}{3}.$$

Spherical Coordinates.

150. If we give to ρ the constant value a, θ and ϕ become the coordinates latitude and longitude of a point upon a spherical surface, and a relation between θ and ϕ becomes the equation of a line drawn upon the spherical surface. For example, a formula of spherical right triangles gives for the equation of the great circle making the angle A with the equator at the zero of longitude

$$\sin\phi = \cot A \tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The product of the differentials of the motions of P corresponding to variations in θ and ϕ, Fig. 24, that is,

$$a^2 \cos\theta\, d\theta\, d\phi,$$

is the element of spherical surface. Hence, for example, to find

the triangular area included between the equator $\theta = 0$, the great circle equation (1), and the meridian $\phi = \alpha$, we have

$$S = a^2 \int_0^{\alpha} \int_0^{\theta} \cos\theta \, d\theta \, d\phi = a^2 \int_0^{\alpha} \sin\theta \, d\phi,$$

where θ is the function of ϕ defined by equation (1). From that equation

$$\sin\theta = \frac{\sin\phi}{\sqrt{(\cot^2 A + \sin^2\phi)}} = \frac{\sin\phi}{\sqrt{(\operatorname{cosec}^2 A - \cos^2\phi)}}.$$

Therefore

$$S = a^2 \int_0^{\alpha} \frac{\sin\phi \, d\phi}{\sqrt{(\operatorname{cosec}^2 A - \cos^2\phi)}} = -a^2 \sin^{-1} \frac{\cos\phi}{\operatorname{cosec} A} \Big]_0^{\alpha}$$

$$= a^2[A - \sin^{-1}(\sin A \cos\alpha)].*$$

Volumes in general.

151. We have seen in Art. 123 that the boundary of the area expressed by a double integral may consist in part of lines whose equations contain only one of the variables, namely, that for which the final integration takes place. But, as explained in Art. 125, in the general case, it is necessary to regard the given area as made up of positive or negative parts of the kind just mentioned. This is done by drawing the loci of constant values of the final variable through the intersections of the lines forming the boundary, or else tangent to them as in Fig. 19. These parts are then expressed by separate integrals.

* If B is the other oblique angle of the right triangle, another trigonometric formula gives

$$\cos B = \sin A \cos\alpha;$$

hence

$$S = a^2(A + B - \tfrac{1}{2}\pi);$$

from which it readily follows that the area of any spherical triangle is equal to $a^2 \times$ the spherical excess in arcual measure.

So also we have seen in Art. 126 that the volume expressed by a triple integral may be bounded in part by a surface whose equation contains only two of the variables, namely, those for which the last two integrations take place. But, in the general case, it is necessary to separate the volume into parts, by means of such surfaces passed through the lines of intersections of the bounding surfaces, or edges of the given solid.

152. Figs. 19 and 20 illustrate this for rectangular coordinates, but similar considerations apply to any other system, and enable us to decide whether it is possible to express a given volume by a single integral. For example, let it be required to find the volume common to the solid of revolution produced by the half-cardioid OAB, Fig. 25, revolving about its axis OB, and the sphere whose centre is at the pole and whose radius is $OC = c$. The volume is the result of integrating the element

$$2\pi r^2 \sin\theta \, dr \, d\theta$$

(found as in Art. 146) over the area OAC. For this area of integration, $\theta = \pi$ and $r = 0$ give one constant limit for each variable, and the others are determined by the equation of the arc OA of the cardioid

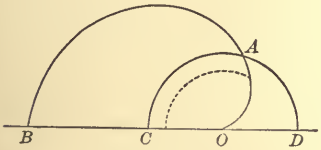

FIG. 25.

$$r = a(1 - \cos\theta), \quad . \quad . \quad (1)$$

and that of the circular arc AC,

$$r = c. \quad . \quad . \quad . \quad (2)$$

Since equation (1) contains both variables, while equation (2) contains r only, we can, by integrating first for θ (and using equation (1) for one of its limits), express the required magnitude by a single integral; thus,

$$V = 2\pi \int_0^c \int_\theta^\pi r^2 \sin\theta \, d\theta \, dr = 2\pi \int_0^c r^2(1 + \cos\theta) dr.$$

Substituting the value of cos θ from equation (1),

$$V = 2\pi \int_0^c r^2 \left(2 - \frac{r}{a}\right) dr = \frac{\pi c^3}{6a}(8a - 3c).$$

If in this example we integrate first for r, it becomes necessary to find θ_1, the value of θ for the point of intersection A, and, regarding the area of integration as separated into two parts by the radius vector OA, to form two integrals, in one of which the upper limit for r is taken from equation (1), and the limits for θ are 0* and θ_1; while in the other r is taken from equation (2), and the limits for θ are θ_1 and π.

153. As a further illustration of these principles, let us resume the consideration of the volume evaluated in Art. 144. The volume is completely enclosed by the spherical and cylindrical surfaces (1) and (2). In the evaluation, after integrating for z, which occurs only in equation (1), the whole volume was represented by a double integral; the limits of the $r\theta$-integration were determined by equation (2) regarded as representing an area in the plane $z = 0$.† In like manner, had we integrated first for θ which occurs only in equation (2), the volume would have been expressed by a single double-integral expression.

But suppose we wished to perform first the integration for r of the element

$$r\,dr\,d\theta\,dz.$$

The indefinite integral is $\frac{1}{2}r^2$, and the lower limit is zero; but the upper limit is, for some values of z and θ, given by equa

*This limit also is in fact determined by the intersection of two sides of the area of integration, namely, that of the curve (1) and $r = 0$ the vanishing inner edge of the area.

† This area happened to be entirely within the "projection" of the sphere (1); that is, the circle $r = a$, within which only z in equation (1) is real. Had this not been the case, the area of integration would have been only that common to the curve (2) and the projection.

tion (1), and for others it is given by equation (2). In other words, the extremity of r moves from the axis of z outward until it reaches either the spherical or the cylindrical surface. Thus the whole field of $z\theta$-integration, which extends from $z = -a$ to $z = a$, and from $\theta = -\frac{1}{2}\pi$ to $\theta = \frac{1}{2}\pi$, must be divided into parts corresponding to these values of the upper limit of r. Taking, as we may by symmetry, one-fourth of this field of integration for one-fourth of the volume V, the dividing line corresponds to the intersection of the sphere and cylinder; it is therefore the result of eliminating r between equations (1) and (2), namely,

$$z = a \sin \theta.$$

Now, when $z < a \sin \theta$, the r of the cylinder is less than that for the sphere, and is therefore to be taken as the upper limit in $\frac{1}{2}r^2\Big]_0^r$; but, when $z > a \sin \theta$, the upper limit must be taken from equation (1). Hence $\frac{1}{4}V = V_1 + V_2$*, where

$$V_1 = \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\int_0^{a \sin \theta} \cos^2 \theta \, dz \, d\theta \quad \text{and} \quad V_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_{a \sin \theta}^{a} (a^2 - z^2) dz \, d\theta.$$

The result will be found to agree with that of Art. 144.

Examples XI.

1. Find, by integrating $rdr\,d\theta$ *first with respect to* θ, the area included by the first whorl of the spiral of Archimedes

$$r = a\theta$$

and the initial line; also, in the same manner, that between the first and second whorls and the initial line. $\frac{4}{3}\pi^3 a^2$; $8\pi^3 a^2$.

* The surface separating V_1 and V_2 is that of a right conoid generated by a line passing through and perpendicular to the axis of z, and also passing through the intersection of the sphere and cylinder. The field of integration may be conceived of as an area lying upon the surface of the cylinder $r = a$. The separating surface traces upon the cylinder the line $z = a \sin \theta$.

2. The paraboloid of revolution

$$x^2 + y^2 = cz$$

is pierced by the right circular cylinder

$$x^2 + y^2 = ax,$$

whose diameter is a, and whose surface contains the axis of the paraboloid; find the volume between the plane of xy and the surfaces of the paraboloid and of the cylinder. $\frac{3\pi a^4}{32c}$.

3. Find the volume cut from a sphere whose radius is a by the cylinder whose base is the curve

$$r = a \cos 3\theta. \qquad \frac{2a^3\pi}{3} - \frac{8a^3}{9}.$$

4. Find the volume cut from a sphere whose radius is a by the cylinder whose base is the curve

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta,$$

supposing $b < a$. $\frac{4\pi a^3}{3} - \frac{16}{9}(a^2 - b^2)^{\frac{3}{2}}$.

5. A right cone, the radius of whose base is a and whose altitude is b, is pierced by a cylinder whose base is a circle having for diameter a radius of the base of the cone; find the volume common to the cone and the cylinder. $\frac{ba^2}{36}(9\pi - 16)$.

6. The axis of a right cone whose semi-vertical angle is α coincides with a diameter of the sphere whose radius is a, the vertex being on the surface of the sphere; find the volume of the portion of the sphere which is outside of the cone. $\frac{4\pi a^3 \cos^4 \alpha}{3}$.

7. Find the volume produced by the revolution of the lemniscate

$$r^2 = a^2 \cos 2\theta$$

about a perpendicular to the initial line. $\frac{\pi^2 a^3 \sqrt{2}}{8}$.

8. Find the volumes generated by the revolution of the large loop and by one of the small loops of the curve

$$r = a \cos\theta \cos 2\theta$$

about a perpendicular to the initial line.

$$\frac{\pi^2 a^3}{16} + \frac{\pi a^3}{5}; \quad \frac{\pi^2 a^3}{32} - \frac{\pi a^3}{10}.$$

9. What is the volume of the cavity which will just permit the cardioid

$$r = a(1 - \cos\theta)$$

to rotate about a line on its plane perpendicular to the initial line?

$$\frac{16 + 5\pi}{4}\pi a^3.$$

10. Find the whole volume enclosed by the surface

$$(x^2 + y^2 + z^2)^3 = a^3 xyz.$$

Transform to the coordinates ρ, ϕ, θ, and show that the solid consists of four equal detached parts. $\frac{a^3}{6}$.

11. Find the volume of that part of the sphere

$$x^2 + y^2 + z^2 = a(x + y + z)$$

for which all three coordinates are positive. $\frac{2\pi + 1}{4}a^3$.

12. In Ex. 11 the coordinate planes separate the sphere into the part A there found, three adjacent parts B, and three parts C. Show that a similar integral to that representing A represents $B - C$, and hence derive the volumes B and C.

$$B = \left[\frac{\pi}{4\sqrt{3}} + \frac{\pi - 2}{12}\right]a^3; \quad C = \left[\frac{\pi}{4\sqrt{3}} - \frac{3\pi - 1}{12}\right]a^3.$$

13. Find the volume generated by the revolution of the curve

$$(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2,$$

in which $a > b$, about the axis of y.

Transform to polar coordinates.

$$\frac{\pi b(2b^2 + 3a^2)}{6} + \frac{\pi a^4}{2\sqrt{(a^2 - b^2)}} \cos^{-1}\frac{b}{a}.$$

14. Find the volume generated by the curve given in the preceding example when revolving about the axis of x.

$$\frac{\pi a(2a^2 + 3b^2)}{6} + \frac{\pi b^4}{2\sqrt{(a^2 - b^2)}} \cdot \log \frac{a + \sqrt{(a^2 - b^2)}}{b}.$$

15. Find the mass of a circular lamina of radius a if the density varies as the distance from a fixed point on the circumference and is μ at the centre. Interpret the integral also as a volume.

$$\frac{32\mu a^2}{9}.$$

16. Find the mass of a square lamina of side $2a$ if the density varies as the distance from the centre and is μ at the middle of a side.

$$\frac{4\mu a^2}{3}[\sqrt{2} + \log(\sqrt{2} + 1)].$$

17. Find the volume between the surface generated by the revolution of the cardioid

$$r = a(1 - \cos\theta)$$

about the initial line and the plane which touches this surface in circle.

$$\frac{\pi a^3}{192}.$$

18. Transform the triple integral element $dx\,dy\,dz$ into the polar element $\rho^2 \cos\theta\, d\rho\, d\theta\, d\phi$. See Arts. 142 and 149.

19. Find the separate values of V_1 and V_2 in Art. 153.

$$V_1 = \frac{a^3}{6}; \quad V_2 = \frac{3\pi - 7}{18}a^3.$$

20. Find the volume cut from a right circular cone of radius a and height a by a plane passing through the centre of the base and parallel to an element of the cone.

$$\frac{3\pi - 4}{18}a^3.$$

XII.

Rectification of Plane Curves.

154. A curve is said to be *rectified* when its length is found in terms of that of some given straight line.

It is shown in Diff. Calc. that, if s denotes the arc of a curve given in rectangular coordinates,

$$ds = \sqrt{(dx^2 + dy^2)}.$$

If, in this expression, the value of dy in terms of x and dx be substituted, the length of an arc in which x varies continuously is found by integration, the limits being the values of x at the extremities.

Thus, in the case of the semi-cubical parabola $ay^2 = x^3$,

$$dy = \frac{3\sqrt{x}\,dx}{2\sqrt{a}}; \quad \text{whence} \quad ds = \frac{\sqrt{(9x + 4a)}}{2\sqrt{a}}dx.$$

To find the arc between the origin (a point of the curve) and the point (x, y), we integrate between zero and x. Thus

$$s = \frac{1}{2\sqrt{a}}\int_0 \sqrt{(9x + 4a)}dx = \frac{1}{27\sqrt{a}}(9x + 4a)^{\frac{3}{2}} - \frac{8a}{27}.$$

155. When x and y are most conveniently expressed as functions of a third variable, the expression for ds in terms of this variable may be used. For example, in the case of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

we put, as in Art. 107,

$$x = a\sin^3\psi,$$

$$y = b\cos^3\psi;$$

whence

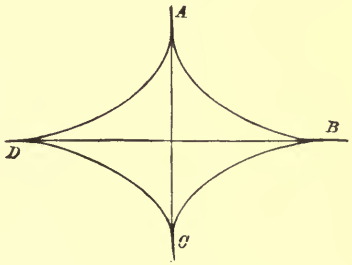

FIG. 26.

$$dx = 3a\sin^2\psi\cos\psi\,d\psi, \quad dy = -3b\cos^2\psi\sin\psi\,d\psi;$$

and therefore

$$ds = 3 \sin\psi \cos\psi \, d\psi \sqrt{(a^2 \sin^2\psi + b^2 \cos^2\psi)}$$
$$= \tfrac{3}{2}\sqrt{[(a^2 - b^2)\sin^2\psi + b^2]}d(\sin^2\psi).$$

The points A and B, Fig. 26, correspond to $\psi = 0$ and $\psi = \frac{1}{2}\pi$ respectively; hence, integrating, we have

$$\text{arc } AB = \frac{(a^2 \sin^2\psi + b^2 \cos^2\psi)^{\frac{3}{2}}}{a^2 - b^2}\Big]_0^{\frac{\pi}{2}} = \frac{a^3 - b^3}{a^2 - b^2} = \frac{a^2 + ab + b^2}{a + b}.$$

Change of the Sign of ds.

156. In the example above, x and y being one-valued functions of ψ, every value of ψ corresponds to a definite point of the curve; and, as ψ varies from 0 to 2π, the point (x, y) describes the whole curve in the direction $ABCD$. But it will be noticed that, $d\psi$ remaining positive, the value of ds becomes zero and changes sign when ψ passes through either of the values 0, $\frac{1}{2}\pi$, π, or $\frac{3}{2}\pi$. This corresponds geometrically to the fact that the point (x, y) stops and reverses the direction of its motion, forming a *stationary point* or *cusp* at each of the points A, B, C and D, as shown in Fig. 26. Such points are thus indicated by a change of sign in ds, and the arcs between them must be considered separately, because the corresponding definite integrals have different signs.

Polar Coordinates.

157. It is proved in Diff. Calc. that in polar coordinates

$$ds = \sqrt{(dr^2 + r^2 \, d\theta^2)}.$$

This is usually expressed in terms of θ. For example, in the case of the cardioid $r = a(1 - \cos\theta) = 2a \sin^2 \frac{1}{2}\theta$, we have

$$dr = 2a \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \, d\theta,$$

whence, by substitution,

$$ds = 2a \sin \tfrac{1}{2}\theta \, d\theta.$$

The limits for the whole perimeter of the curve are 0 and 2π, and ds remains positive for the whole interval. Therefore

$$s = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = -4a\cos\frac{\theta}{2}\Big]_0^{2\pi} = 8a.$$

Rectification of Curves of Double Curvature.

158. Let σ denote the length of the arc of *a curve of double curvature*; that is, one which does not lie in a plane, and suppose the curve to be referred to rectangular coordinates x, y and z. If at any point of the curve the differentials of the coordinates be drawn in the directions of their respective axes, a rectangular parallelopiped will be formed, whose sides are dx, dy and dz, and whose diagonal is $d\sigma$. Hence

$$d\sigma = \sqrt{(dx^2 + dy^2 + dz^2)}.$$

The curve is determined by means of two equations connecting x, y and z, one of which usually expresses the value of y in terms of x, and the other that of z in terms of x. We can then express $d\sigma$ in terms of x and dx.

If the given equations contain all the variables, equations of the required form may be obtained by elimination.

159. An equation containing the two variables x and y only is evidently the equation of *the projection upon the plane of xy* of a curve traced upon the surface determined by the other equation. Let s denote the length of this projection: then, since $ds^2 = dx^2 + dy^2$,

$$d\sigma = \sqrt{(ds^2 + dz^2)},$$

in which ds may, if convenient, be expressed in polar coordinates; thus,

$$d\sigma = \sqrt{(dr^2 + r^2\,d\theta^2 + dz^2)}.$$

160. As an illustration, let us use this formula to determine the length of the loxodromic curve from the equation of the sphere,

$$x^2 + y^2 + z^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

upon which it is traced, and its projection upon the plane of the equator, of which the equation is

$$2a = \sqrt{(x^2 + y^2)}\left(e^{n \tan^{-1}\frac{y}{x}} + e^{-n \tan^{-1}\frac{y}{x}}\right),$$

or in polar coordinates

$$2a = r\,(e^{n\theta} + e^{-n\theta}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) is equivalent to

$$r^2 + z^2 = a^2;$$

and, denoting the latitude of the projected point by ϕ, this gives

$$z = a \sin \phi, \qquad r = a \cos \phi. \quad . \quad . \quad . \quad (3)$$

In order to express $d\theta$ in terms of ϕ, we substitute the value of r in (2); whence

$$e^{n\theta} + e^{-n\theta} = 2 \sec \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and by differentiation

$$e^{n\theta} - e^{-n\theta} = \frac{2}{n} \sec \phi \tan \phi \frac{d\phi}{d\theta}. \quad . \quad . \quad . \quad . \quad (5)$$

Squaring and subtracting,

$$4 = \frac{4 \sec^2 \phi}{n^2}\left[n^2 - \tan^2 \phi \frac{d\phi^2}{d\theta^2}\right],$$

which reduces to

$$d\theta^2 = \frac{\sec^2 \phi \, d\phi^2}{n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From equations (3) and (6)

$$r^2 d\theta^2 = \frac{a^2}{n^2} d\phi^2,$$

$$dr^2 = a^2 \sin^2 \phi \, d\phi^2,$$
$$dz^2 = a^2 \cos^2 \phi \, d\phi^2;$$

whence substituting in the value of $d\sigma$ (p. 191)

$$d\sigma = a \sqrt{\left(1 + \frac{1}{n^2}\right)} d\phi.$$

Integrating,

$$\sigma = a \frac{\sqrt{(n^2 + 1)}}{n} \int_\alpha^\beta d\phi = a \frac{\sqrt{(n^2 + 1)}}{n} (\beta - \alpha),$$

where α and β denote the latitudes of the extremities of the arc.

Examples XII.

1. Find the length of an arc measured from the vertex of the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

and show that the area between the coordinate axes and any arc is proportional to the arc.

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

$$A = cs.$$

2. Find the length of an arc measured from the vertex of the parabola

$$y^2 = 4ax.$$

$$\sqrt{(ax + x^2)} + a \log \frac{\sqrt{x} + \sqrt{(x + a)}}{\sqrt{a}}.$$

3. Find the length of the curve

$$e^y = \frac{e^x + 1}{e^x - 1},$$

between the points whose abscissas are a and b.

$$\log \frac{e^{2b} - 1}{e^{2a} - 1} + a - b.$$

4. Find the length, measured from the origin, of the curve

$$y = a \log \frac{a^2 - x^2}{a^2}.$$

$$a \log \frac{a + x}{a - x} - x.$$

5. Given the differential equation of the tractrix,

$$\frac{dy}{dx} = - \frac{y}{\sqrt{(a^2 - y^2)}},$$

and, assuming (o, a) to be a point of the curve, find the value of s as measured from this point, and also the value of x in terms of y; that is, find the rectangular equation of the curve.

$$s = a \log \frac{y}{a}.$$

$$x = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

6. Find the length of one branch of the cycloid

$$x = a(\psi - \sin \psi), \qquad y = a(1 - \cos \psi).$$

$$8a.$$

7. When the cycloid is referred to its vertex, the equations being

$$x = a(1 - \cos \psi), \qquad y = a(\psi + \sin \psi),$$

prove that $s = \sqrt{(8ax)}$.

8. Find the length from the point $(a, 0)$ of the curve

$$x = 2a\cos\psi - a\cos 2\psi,$$

$$y = 2a\sin\psi - a\sin 2\psi.$$

$8a\,(1 - \cos\frac{1}{2}\psi)$.

9. Show that the curve,

$$x = 3a\cos\psi - 2a\cos^3\psi, \qquad y = 2a\sin^3\psi$$

has cusps at the points given by $\psi = 0$ and $\psi = \pi$; and find the whole length of the curve. $12a$.

10. Find the length of the arc of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} = 1$$

between the points where it touches the axes.

$$\frac{a^3 + b^3}{a^2 + b^2} + \frac{a^2b^2}{(a^2 + b^2)^{\frac{3}{2}}}\log\frac{[\sqrt{(a^2 + b^2)} + a][\sqrt{(a^2 + b^2)} + b]}{ab}.$$

11. Show that the curve

$$x = 2a\cos^2\theta\,(3 - 2\cos^2\theta), \qquad y = 4a\sin\theta\cos^3\theta$$

has three cusps, and that the length of each branch is $\frac{8a}{3}$.

12. Find the length of the arc between the points at which the curve

$$x = a\cos^2\theta\cos 2\theta, \qquad y = a\sin^2\theta\sin 2\theta$$

cuts the axes. $\frac{2 + \sqrt{2}}{3}a$.

13. Show that the curve

$$x = a\cos\psi\,(1 + \sin^2\psi),$$
$$y = a\sin\psi\cos^2\psi$$

is symmetrical to the axes, and find the length of the arcs between the cusps.

$$a\left(\sqrt{2} - \sin^{-1}\frac{1}{\sqrt{3}}\right);$$

$$a\left(\sqrt{2} + \cos^{-1}\frac{1}{\sqrt{3}}\right).$$

14. Find the length of one branch of the epicycloid

$$x = (a + b)\cos\psi - b\cos\frac{a + b}{b}\psi,$$

$$y = (a + b)\sin\psi - b\sin\frac{a + b}{b}\psi.$$

$$\frac{8b\,(a + b)}{a}.$$

15. Show that the curve

$$x = 9a\sin\psi - 4a\sin^3\psi,$$
$$y = -3a\cos\psi + 4a\cos^3\psi$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs, (α) between the double points, (β) between a double point and a cusp, and (γ) the arc connecting two cusps, and not passing through the double points.

$$(\alpha),\ a\,(\pi + 3\sqrt{3});$$

$$(\beta),\ \frac{\pi a}{2};$$

$$(\gamma),\ a\,(3\sqrt{3} - \pi).$$

16. Find the whole length of the curve

$$x = 3a\sin\psi - a\sin^3\psi,$$
$$y = a\cos^3\psi.$$

$3\pi a$.

17. Find the length, measured from the pole, of any arc of the equiangular spiral

$$r = a\varepsilon^{n\theta},$$

in which $n = \cot \alpha$. $r \sec \alpha$.

18. Prove by integration that the arc subtending the angle θ at the circumference in a circle whose radius is a, is $2a\theta$.

19. Find the length, measured from the origin, of the curve defined by the equations

$$y = \frac{x^2}{2a}, \qquad z = \frac{x^3}{6a^2}.$$

$x + \frac{x^3}{6a^2}$.

20. Find the length, measured from the origin, of the intersection of the surfaces

$$y = 4n \sin x, \qquad z = 2n^2 (2x + \sin 2x).$$

$(4n^2 + 1)x + 2n^2 \sin 2x$.

21. Find the length, measured from the origin, of the intersection of the cylindrical surfaces

$$(y - x)^2 = 4ax, \qquad 9a (z - x)^2 = 4x^3.$$

$\frac{2x^{\frac{3}{2}}}{3\sqrt{a}} + 2\sqrt{(ax)} + x$.

22. If upon the hyperbolic cylinder

$$\frac{y^2}{c^2} - \frac{z^2}{b^2} = 1,$$

a curve whose projection upon the plane of xy is the catenary

$$y = \frac{c}{2}\left(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}}\right)$$

be traced, prove that any arc of the curve bears to the corresponding arc of its projection the constant ratio $\sqrt{(b^2 + c^2)} : c$.

XIII.

Surfaces of Solids of Revolution.

161. The surface of a solid of revolution may be generated by the circumference of the circular section made by a plane perpendicular to the axis of revolution. Thus in Fig. 27, the surface produced by the revolution of the curve AB about the axis of x is regarded as generated by the circumference PQ. The radius of this circumference is y, and its *plane* has a motion whose differential is dx, but every point in the circumference itself has a motion whose differential is ds, s denoting an arc of the curve AB. Hence, denoting the required surface by S, we have

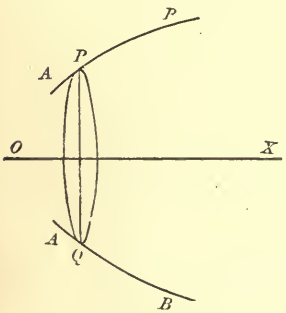

FIG. 27.

$$dS = 2\pi y\, ds = 2\pi y \sqrt{(dx^2 + dy^2)}.$$

The value of dS must of course be expressed in terms of a single variable before integration.

162. For example, let us determine the area of the zone of spherical surface included between any two parallel planes. The radius of the sphere being a, the equation of the revolving curve is

$$x^2 + y^2 = a^2;$$

whence

$$y = \sqrt{(a^2 - x^2)},$$

$$dy = -\frac{x\,dx}{\sqrt{(a^2 - x^2)}},$$

$$ds = \frac{a\,dx}{\sqrt{(a^2 - x^2)}},$$

and

$$dS = 2\pi a\, dx;$$

therefore

$$S = 2\pi a \int dx = 2\pi a\,(x_2 - x_1).$$

Since $x_2 - x_1$ is the distance between the parallel planes, the area of a zone is the product of its altitude by $2\pi a$, the circumference of a great circle, and the area of the whole surface of the sphere is $4\pi a^2$.

163. When the curve is given in polar coordinates, it is convenient to transform the expression for S to polar coordinates. Thus, if the curve revolves about the initial line,

$$S = 2\pi \int y\, ds = 2\pi \int r \sin\theta \sqrt{(dr^2 + r^2\, d\theta^2)}.$$

For example, if the curve is the cardioid

$$r = 2a \sin^2 \frac{1}{2}\theta,$$

we find, as in Art. 157,

$$ds = 2a \sin \frac{1}{2}\theta\, d\theta.$$

Hence

$$S = 16\pi a^2 \int_0^{\pi} \sin^4 \frac{1}{2}\theta \cos \frac{1}{2}\theta\, d\theta$$

$$= \frac{32\pi a^2}{5} \sin^5 \frac{1}{2}\theta \Big]_0^{\pi} = \frac{32\pi a^2}{5}.$$

Areas of Surfaces in General.

164. Let a surface be referred to rectangular coordinates x, y and z; the projection of a given portion of the surface upon the plane of xy is a plane area determined by a given relation between x and y. We may take as the elements of the surface the portions which are projected upon the corresponding

elements of area in the plane of xy. If at a point within the element of surface, which is projected upon a given element $\Delta x\, \Delta y$, a tangent plane be passed, and if γ denote the inclination of this plane to the plane of xy, the area of the corresponding element *in the tangent plane* is

$$\sec \gamma\, \Delta x\, \Delta y.$$

The surface is evidently the limit of the sum of the elements in the tangent planes when Δx and Δy are indefinitely diminished. Now $\sec \gamma$ is a function of the coordinates of the point of contact of the tangent plane; and since these coordinates are values of x and y which lie respectively between x and $x + \Delta x$ and between y and $y + \Delta y$, it follows, as in Art. 122, that this limit is

$$S = \int\int \sec \gamma\, dx\, dy.$$

165. The value of $\sec \gamma$ may be derived by the following method. Through the point P of the surface let planes be passed parallel to the coordinate planes, and let PD, and PE, Fig. 28, be the intersections of the tangent plane with the planes parallel to the planes of xz and yz. Then PD and PE are tangents at P to the sections of the surface made by these planes. The equations of these sections are found by regarding y and x in turn as constants in the equation of the surface; therefore, denoting the inclinations of these tangent lines to the plane of xy by ϕ and ψ, we have

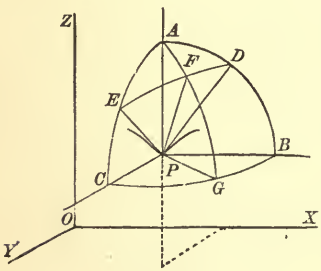

FIG. 28.

$$\tan \phi = \frac{dz}{dx}, \qquad \text{and} \qquad \tan \psi = \frac{dz}{dy},$$

in which $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are partial derivatives derived from the equation of the surface.

If the planes be intersected by a spherical surface whose centre is P, ADE is a spherical triangle right angled at A, whose sides are the complements of ϕ and ψ. Moreover, if a plane perpendicular to the tangent plane PED be passed through AP, the angle FPG will be γ, and the perpendicular from the right angle to the base of the triangle the complement of γ.

Denoting the angle EAF by θ, the formulæ for solving spherical right triangles give

$$\cos\theta = \frac{\tan\psi}{\tan\gamma}, \qquad \text{and} \qquad \sin\theta = \frac{\tan\phi}{\tan\gamma}.$$

Squaring and adding,

$$1 = \frac{\tan^2\psi + \tan^2\phi}{\tan^2\gamma},$$

$$\tan^2\gamma = \tan^2\psi + \tan^2\phi;$$

whence $$\sec^2\gamma = 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2.$$

Substituting in the formula derived in Art. 164, we have

$$S = \iint \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]}\, dx\, dy.$$

166. It is sometimes more convenient to employ the polar

element of the projected area. Thus the formula becomes

$$S = \iint \sec \gamma \, r \, dr \, d\theta,$$

where sec γ has the same meaning as before.

For example, let it be required to find the area of the surface of a hemisphere intercepted by a right cylinder having a radius of the hemisphere for one of its diameters. From the equation of the sphere,

$$x^2 + y^2 + z^2 = a^2, \quad \ldots\ldots\ldots \quad (1)$$

we derive

$$\frac{dz}{dx} = -\frac{x}{z}, \qquad \frac{dz}{dy} = -\frac{y}{z};$$

whence

$$\sec \gamma = \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} = \frac{a}{z};$$

therefore

$$S = a \iint \frac{r \, dr \, d\theta}{z},$$

the integration extending over the area of the circle

$$r = a \cos \theta. \quad \ldots\ldots\ldots \quad (2)$$

Since equation (1) is equivalent to

$$z^2 + r^2 = a^2,$$

$$S = a \int\int_{r_1}^{r_2} \frac{r \, dr}{\sqrt{(a^2 - r^2)}} \, d\theta = a \int [\sqrt{(a^2 - r_1^2)} - \sqrt{(a^2 - r_2^2)}] \, d\theta.$$

From (2) the limits for r are $r_1 = 0$, and $r_2 = a \cos \theta$, hence

$$S = a^2 \int (1 - \sin \theta)\, d\theta,$$

in which $a \sin \theta$ is put for the *positive* quantity $\sqrt{(a^2 - r_2^2)}$. The limits for θ are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, but since $\sin \theta$ is in this case to be regarded as invariable in sign, we must write

$$S = 2a^2 \int_0^{\frac{\pi}{2}} (1 - \sin \theta)\, d\theta = \pi a^2 - 2a^2.$$

If another cylinder be constructed, having the opposite radius of the hemisphere for diameter, the surface removed is $2\pi a^2 - 4a^2$, and the surface which remains is $4a^2$, a quantity commensurable with the square of the radius. This problem was proposed in 1692, in the form of an enigma, by Viviani, a Florentine mathematician.

Examples XIII.

1. Find the surface of the paraboloid whose altitude is a, and the radius of whose base is b.

$$\frac{\pi b}{6a^2}\left[(4a^2 + b^2)^{\frac{3}{2}} - b^3\right].$$

2. Prove that the surface generated by the arc of the catenary given in Ex. XII., 1, revolving about the axis of x, is equal to

$$\pi(cx + sy).$$

3. Find the whole surface of the oblate spheroid produced by the

revolution of an ellipse about its minor axis, a denoting the major, b the minor semi-axis, and e the excentricity, $\frac{\sqrt{(a^2 - b^2)}}{a}$.

$$2\pi a^2 + \pi \frac{b^2}{e} \log \frac{1 + e}{1 - e}.$$

4. Find the whole surface of the prolate spheroid produced by the revolution of the ellipse about its major axis, using the same notation as in Ex. 3.

$$2\pi b^2 + 2\pi ab \frac{\sin^{-1} e}{e}.$$

5. Find the surface generated by the cycloid

$$x = a(\psi - \sin\psi), \qquad y = a(1 - \cos\psi)$$

revolving about its base. $\frac{64}{3}\pi a^2$.

6. Find the surface generated when the cycloid revolves about the tangent at its vertex.

$$\frac{32}{3}\pi a^2.$$

7. Find the surface generated when the cycloid revolves about its axis.

$$8\pi a^2\left(\pi - \frac{4}{3}\right).$$

8. Find the surface generated by the revolution of one branch of the tractrix (see Ex. XII., 5) about its asymptote.

$2\pi a^2$.

9. Find the surface generated by the revolution about the axis of x of the portion of the curve

$$y = e^x$$

which is on the left of the axis of y.

$$\pi[\sqrt{2} + \log(1 + \sqrt{2})].$$

10. Find the surface generated by the revolution about the axis of x of the arc between the points for which $x = a$ and $x = b$ in the hyperbola

$$xy = k^2.$$

$$\pi k^2\left[\log\frac{b^2 + \sqrt{(k^4 + b^4)}}{a^2 + \sqrt{(k^4 + a^4)}} + \frac{\sqrt{(k^4 + a^4)}}{a^2} - \frac{\sqrt{(k^4 + b^4)}}{b^2}\right]$$

11. Show that the surface of a cylinder whose generating lines are parallel to the axis of z is represented by the integral

$$S = \int z\, ds,$$

where s denotes the arc of the base in the plane of xy. Hence, deduce the surface cut from a right circular cylinder whose radius is a, by a plane passing through the centre and making the angle α with the plane of the base. $2a^2 \tan \alpha$.

12. Find the surface of that portion of the cylinder in the problem solved in Art. 166, which is within the hemisphere. $2a^2$.

13. Find the surface of a circular spindle, a being the radius and $2c$ the chord.

$$4\pi a\left[c - \sqrt{(a^2 - c^2)}\sin^{-1}\frac{c}{a}\right].$$

XIV.

The Area generated by a Straight Line moving in any Manner in a Plane.

167. If a straight line of indefinite length moves in any manner whatever in a plane, there is at each instant a point of the line about which it may be regarded as rotating. This point we shall call *the centre of rotation* for the instant. The rate of motion of every point of the line in a direction perpendicular to the line itself is at the instant the same as it would be if the line were rotating at the same angular rate about this point as a fixed centre.* Hence it follows that the area generated by a definite portion of the line has at the instant the same rate as if the line were rotating about a fixed instead of a variable centre.

168. Suppose at first that the centre of rotation is on the generating line produced, ρ_1 and ρ_2 denoting the distances from the centre of the extremities of the generating line, and let ϕ denote its inclination to a fixed line. By substitution in the general formula derived in Art. 110, we have

$$dA = \frac{1}{2}(\rho_2^2 - \rho_1^2)\, d\phi.$$

* Compare Diff. Calc., Art. 326 [Abridged Ed., Art. 176], where the moving line is the normal to a given curve, and the centre of rotation is the centre of curvature of the given curve. If the line is moving without change of direction, the centre is of course at an infinite distance.

When the line is regarded as forming a part of a rigidly connected system in motion, its centre of rotation is the foot of a perpendicular dropped upon it from the *instantaneous centre* of the motion of the system. Thus, if the tangent and normal in the illustration cited are rigidly connected, the centre of curvature, C, is the *instantaneous centre* of the motion of the system, and the point of contact, P, is the centre of rotation for the tangent line.

Applications.

169. The area between a curve and its evolute may be generated by the radius of curvature ρ, whose inclination to the axis of x is $\phi + \frac{1}{2}\pi$, in which ϕ denotes the inclination of the tangent line. Since the centre of rotation is one extremity of the generating line ρ, the differential of this area is found by substituting in the general expression $\rho_1 = 0$ and $\rho_2 = \rho$. Hence when ρ is expressed in terms of ϕ,

$$A = \frac{1}{2}\int \rho^2 \, d\phi$$

expresses the area between an arc of a given curve, its evolute, and the radii of curvature of its extremities, the limits being the values of ϕ at the ends of the given arc.

170. For example, in the case of the cardioid

$$r = a(1 - \cos\theta),$$

it is readily shown, from the results obtained in Art. 157, that the angle between the tangent and the radius vector is $\frac{1}{2}\theta$; and therefore $\phi = \frac{3}{2}\theta$, and

$$\rho = \frac{ds}{d\phi} = \frac{4a}{3}\sin\frac{\phi}{3}.$$

To obtain the whole area between the curve and its evolute, the limits for θ are 0 and 2π; hence the limits for ϕ are 0 and 3π. Therefore

$$A = \frac{1}{2}\int_0^{3\pi} \rho^2 \, d\phi = \frac{8a^2}{9}\int_0^{3\pi} \sin^2\frac{\phi}{3} \, d\phi = \frac{4\pi a^2}{3}.$$

171. As another application of the general formula of Art. 168, let one end of a line of fixed length a be moved

along a given line in a horizontal plane, while a weight attached to the other extremity is drawn over the plane by the line, and is therefore always moving in the direction of the line itself. The line of fixed length in this case turns about the weight as a moving centre of rotation. Hence the area generated while the line turns through a given angle is the same as that of the corresponding sector of a circle whose radius is a.

The curve described by the weight is called *a tractrix*, and the line along which the other extremity is moved is *the directrix*. When the axis of x is the directrix, and the weight starts from the point $(0, a)$, the common tractrix is described; hence the area between this curve and the axes is $\frac{1}{4}\pi a^2$.

172. Again, in the generation of the cycloid, Diff. Calc., Art. 278 [Abridged Ed., Art. 156], the variable chord RP may be regarded as generating the area. The point R has a motion in the direction of the tangent RX; the point P partakes of this motion, which is the motion of the centre C, and also has an equal motion, due to the rotation of the circle in the direction of the tangent to the circle at P. Since the tangents at P and R are equally inclined to PR, the motion of P in a direction perpendicular to PR is double the component, in this direction, of the motion of R. Therefore the centre of rotation of PR is beyond R at a distance from it equal to PR. Hence, denoting PRO by ϕ,

$$\rho_1 = PR = 2a \sin \phi, \qquad \rho_2 = 2PR = 4a \sin \phi.$$

Substituting in the formula of Art. 168, we have for the area of the cycloid, since PRO varies from 0 to π,

$$A = 6a^2 \int_0^{\pi} \sin^2 \phi \, d\phi = 3\pi a^2.$$

Sign of the Generated Area.

173. Let AB be the generating line, and C the centre of rotation. The expression,

$$dA = \tfrac{1}{2}(\rho_2^2 - \rho_1^2)\, d\phi, \quad \ldots \ldots \quad (1)$$

for the differential of the area, was obtained upon the supposition that A and B were on the same side of C. Then supposing $\rho_2 > \rho_1$, and that the line rotates in the positive direction, as in figure 29, *the differential of the area is positive;* and we notice that every point in the area generated is swept over by the line AB, *the left hand side as we face in the direction AB preceding.*

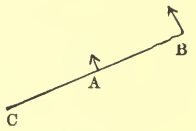

FIG. 29.

174. We shall now show that in every case, the formula requires that an area swept over with the left side preceding, shall be considered as positively generated, and one swept over in the opposite direction as negatively generated.

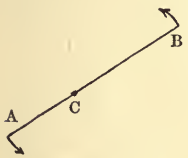

FIG. 30.

In the first place, if C is between A and B so that ρ_1 is negative, as in figure 30, ρ_1^2 is still positive, and formula (1) still gives the difference between the areas generated by CB and AC. Hence the latter area, which is now generated by a part of the line AB, must be regarded as generated negatively, but the *right hand side* as we face in the direction AB of this portion of the line is now preceding, which agrees with the rule given in Art. 173.

Again, if C is beyond B, the formula gives the difference of the generated areas; but since ρ_1^2 is numerically greater than ρ_2^2, in this case, dA is negative, and the area generated by AB is the difference of the areas, and is negative by the rule.

Finally, if the direction of rotation be reversed, $d\phi$ and therefore dA change sign, but the opposite side of each portion of the line becomes in this case the preceding side.

175. We may now put the expression for the area in another form. For

$$dA = \frac{1}{2}(\rho_2^2 - \rho_1^2)\, d\phi = (\rho_2 - \rho_1)\, \frac{\rho_2 + \rho_1}{2}\, d\phi;$$

whatever be the signs of ρ_2 and ρ_1, the first factor is the length of AB, which we shall denote by l, and the second factor is the distance of the middle point of AB from the centre of rotation, which we shall denote by ρ_m. Hence, putting

$$\rho_2 - \rho_1 = l, \qquad \text{and} \qquad \frac{\rho_2 + \rho_1}{2} = \rho_m,$$

we have

$$A = \int l \rho_m \, d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $\rho_m\, d\phi$ is the differential of the motion of the middle point in a direction perpendicular to AB, this expression shows that the differential of the area is the product of this differential by the length of the generating line.

Areas generated by Lines whose Extremities describe Closed Circuits.

176. Let us now suppose the generating line AB to move from a given position, and to return to the same position, each of the extremities A and B describing a closed curve in the positive direction, as indicated by the arrows in figure 31. It is readily seen that every point which is in the area described by B, and not in that described by A, will be swept over at least once by the line AB, the left side preceding, and if passed over more than once, there will be

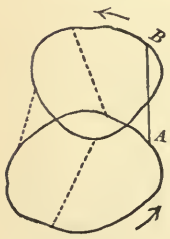

FIG. 31.

an excess of one passage, the left side preceding. Therefore the area within the curve described by B, and not within that described by A, will be generated positively. In like manner the area within the curve described by A, and not within that described by B, will be generated negatively. Furthermore, all points within both or neither of these curves are passed over, if at all, an equal number of times in each direction, so that the area common to the two curves and exterior to both disappears from the expression for the area generated by AB.

Hence it follows that, *regarding a closed area whose perimeter is described in the positive direction as positive, the area generated by a line returning to its original position is the difference of the areas described by its extremities.* This theorem is evidently true generally, if areas described in the opposite direction are regarded as negative.

Amsler's Planimeter.

177. The theorem established in the preceding article may be used to demonstrate the correctness of the method by which an area is measured by means of the *Polar Planimeter*, invented by Professor Amsler, of Schaffhausen.

This instrument consists of two bars, OA and AB, Fig. 32, jointed together at A. The rod OA turns on a fixed pivot at O, while a tracer at B is carried in the positive direction completely around the perimeter of the area to be measured. At some point C of the bar AB a small wheel is fixed, having its axis parallel to AB, and its circumference resting upon the paper. When B is moved, this wheel has a sliding and a rolling motion; the latter motion is recorded by an attachment by means of which the number of turns and parts of a turn of the wheel are registered.

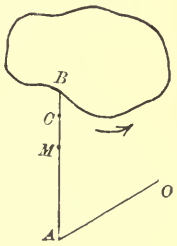

FIG. 32.

178. Let M be the middle point of AB, and let

$$OA = a, \qquad AB = b, \qquad MC = c.$$

Since b is constant, the area described by AB is by equation (2), Art. 175,

$$\text{Area } AB = b\int \rho_m \, d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Denoting the linear distance registered on the circumference of the wheel by s, ds is the differential of the motion of the point C, in a direction perpendicular to AB, and since the distance of this point from the centre of rotation is $\rho_m + c$,

$$ds = (\rho_m + c)\, d\phi:$$

substituting in (1) the value of $\rho_m \, d\phi$,

$$\text{Area } AB = b\int ds - bc\int d\phi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

179. Two cases arise in the use of the instrument. When, as represented in Fig. 32, O is outside the area to be measured, the point A describes no area, and by the theorem of Art. 176, equation (2) represents simply the area described by B. In this case ϕ returns to its original value, hence $\int d\phi$ vanishes, and denoting the area to be measured by A, equation (2) becomes

$$A = bs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the second case, when O is within the curve traced by B, the point A describes a circle whose area is πa^2, and the limit-

ing values of ϕ differ by a complete revolution. Hence in this case equation (2) becomes

$$A + \pi a^2 = bs - 2\pi bc,$$

or

$$A = bs + \pi (a^2 - 2bc).* \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In another form of the planimeter the point A moves in a straight line, and the same demonstration shows that the area is always equal to bs.

Examples XIV.

1. The involute of a circle whose radius is a is drawn, and a tangent is drawn at the opposite end of the diameter which passes through the cusp; find the area between the tangent and the involute.

$$\frac{a^2\pi (3 + \pi^2)}{3}.$$

2. Two radii vectores of a closed oval are drawn from a fixed point within, one of which is parallel to the tangent at the extremity of the other; if the parallelogram be completed, the area of the locus of its vertex is double the area of the given oval.

3. Show that the area of the locus of the middle point of the chord joining the extremities of the radii vectores in Ex. 2, is one half the area of the given oval.

---

* The planimeter is usually so constructed that the positive direction of rotation is with the hands of a watch. The bar b is adjustable, but the distance AC is fixed so that c varies with b. Denoting AC by q, we have $c = q - \frac{1}{2}b$, and the constant to be added becomes $C = \pi (a^2 - 2bq + b^2)$ in which a and q are fixed and b adjustable. In some instruments q is negative.

It is to be noticed that in the second case s may be negative; the area is then the numerical difference between the constant and bs.

4. Prove that the difference of the perimeters of two parallel ovals, whose distance is b, is $2\pi b$, and that the difference of their areas is the product of b and the half sum of their perimeters.

5. From a fixed point on the circumference of a circle whose radius is a a radius vector is drawn, and a distance b is measured from the circumference upon the radius vector produced; the extremity of b therefore describes a limaçon: show that the area generated by b when $b > 2a$ is the area of the limaçon diminished by twice the area of the circle, and thence determine the area of the limaçon.

$$\pi(2a^2 + b^2).$$

6. Verify equation (4), Art. 179, when the tracer describes the circle whose radius is $a + b$.

7. Verify the value of the constant in equation (4), Art. 179, by determining the circle which may be described by the tracer without motion of the wheel.

8. If, in the motion of a crank and connecting rod (the line of motion of the piston passing through the centre of the crank), Amsler's recording wheel be attached to the connecting rod at the piston end, determine s geometrically, and verify by means of the area described by the other end of the rod.

9. The length of the crank in Ex. 8 being a, and that of the connecting rod b, find the area of the locus of a point on the connecting rod at a distance c from the piston end.

$$\frac{\pi a^2 c}{b}.$$

10. If a line AB of fixed length move in a plane, returning to its original position *without making a complete revolution*, denoting the areas of the curves described by its extremities by (A) and (B), determine the area of the curve described by a point cutting AB in the ratio $m : n$.

$$\frac{n(A) + m(B)}{m + n}$$

11. If the line in Ex. 10 return to its original position after *making a complete revolution*, prove *Holditch's Theorem*; namely, that the area of the curve described by a point at the distance c and c' from A and B is

$$\frac{c'(A) + c(B)}{c + c'} - \pi cc'.$$

12. Show by means of Ex. 11 that, if a chord of fixed length move around an oval, and a curve be described by a point at the distances c and c' from its ends, the area between the curves will be $\pi cc'$.

XV.

Approximate Expressions for Areas and Volumes.

180. When the equation of a curve is unknown, the area between the curve, the axis of x, and two ordinates may be approximately expressed in terms of the base and a limited number of ordinates, which are supposed to have been measured.

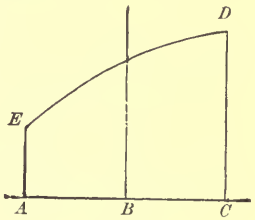

FIG. 33.

Let $ABCDE$ be the area to be determined; denote the length of the base by $2h$; and let the ordinates at the extremities and middle point of the base be measured and denoted by y_1, y_2, and y_3. Taking the base for the axis of x, and the middle point as origin, let it be assumed that the curve has an equation of the form

$$y = A + Bx + Cx^2 + Dx^3; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then the area required is

$$A = \int_{-h}^{h} y\,dx = Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} + \frac{Dx^4}{4}\Big]_{-h}^{h} = \frac{h}{3}(6A + 2Ch^2), \quad . \quad (2)$$

in which which A and C are unknown.

In order to express the area in terms of the measured ordinates, we have from equation (1),

$$y_1 = A + Bh + Ch^2 + Dh^3,$$
$$y_2 = A,$$
$$y_3 = A - Bh + Ch^2 - Dh^3;$$

whence we derive

$$y_1 + y_3 = 2A + 2Ch^2,$$
$$y_1 + 4y_2 + y_3 = 6A + 2Ch^2;$$

and substituting in (2),

$$A = \frac{h}{3}(y_1 + 4y_2 + y_3).$$

It will be noticed that this formula gives a perfectly accurate result when the curve is really a parabolic curve of the third or a lower degree.

181. If the base be divided into three equal intervals, each denoted by h, and the ordinates at the extremities and at the points of division measured, we have, by assuming the same equation,

$$A = \int_{-\frac{3h}{2}}^{\frac{3h}{2}} y\,dx = \frac{3h}{4}(4A + 3Ch^2) \quad \ldots\ldots \quad (1)$$

From the equation of the curve,

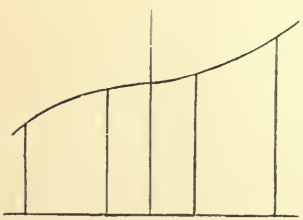

FIG. 34.

$$y_1 = A - \frac{3Bh}{2} + \frac{9Ch^2}{4} - \frac{27Dh^3}{8},$$
$$y_2 = A - \frac{Bh}{2} + \frac{Ch^2}{4} - \frac{Dh^3}{8},$$
$$y_3 = A + \frac{Bh}{2} + \frac{Ch^2}{4} + \frac{Dh^3}{8},$$
$$y_4 = A + \frac{3Bh}{2} + \frac{9Ch^2}{4} + \frac{27Dh^3}{8};$$

whence $$y_1 + y_4 = 2A + \frac{9Ch^2}{2},$$

$$y_2 + y_3 = 2A + \frac{Ch^2}{2}.$$

From these equations we obtain

$$A = \frac{-y_1 + 9y_2 + 9y_3 - y_4}{16},$$

and $$Ch^2 = \frac{y_1 - y_2 - y_3 + y_4}{4}.$$

Substituting in equation (1),

$$A = \frac{3h}{8}(y_1 + 3y_2 + 3y_3 + y_4).$$

Simpson's Rules.

182. The formulæ derived in Articles 180 and 181, although they were first given by Cotes and Newton, are usually known as *Simpson's Rules*, the following extensions of the formulæ having been published in 1743, in his *Mathematical Dissertations.*

If the whole base be divided into an even number n of parts, each equal to h, and the ordinates at the points of division be numbered in order from end to end, then by applying the first formula to the areas between the alternate ordinates, we have

$$A = \frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 \cdots + 4y_n + y_{n+1}).$$

That is to say, the area is equal to the product of the sum of the extreme ordinates, four times the sum of the even-num-

bered ordinates, and twice the sum of the remaining odd-numbered ordinates, multiplied by one third of the common interval.

Again, if the base be divided into a number of parts divisible by three, we have, by applying the formula derived in Art. 181 to the areas between the ordinates $y_1 y_4$, $y_4 y_7$, and so on,

$$A = \frac{3h}{8}(y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 \cdots + 3y_n + y_{n+1}).$$

Cotes' Method of Approximation.

183. The method employed in Articles 180 and 181 is known as *Cotes' Method.* It consists in assuming the given curve to be a parabolic curve of the highest order which can be made to pass through the extremities of a series of equidistant measured ordinates.

The equation of the parabolic curve of the nth order contains $n + 1$ unknown constants; hence, in order to eliminate these constants from the expression for an area defined by the curve, it is in general necessary to have $n + 1$ equations connecting them with the measured ordinates. Hence, if n denote the number of intervals between measured ordinates over which the curve extends, the curve will in general be of the nth degree.*

* If H denotes the whole base, the first factor is always equivalent to H divided by the sum of the coefficients of the ordinates; for if all the ordinates are made equal, the expression must reduce to Hy_1. Thus, each of the rules for an approximate area, including those derived by repeated applications, as in Art. 182, may be regarded as giving an expression for the *mean ordinate.* The coefficients of the ordinates, according to Cotes' method, for all values of n up to $n = 10$, may be found in Bertrand's *Calcul Intégral*, pages 333 and 334. For example (using detached coefficients for brevity), we have, when $n = 4$,

$$A = \frac{H}{90}[7, 32, 12, 32, 7];$$

and when $n = 6$,

$$A = \frac{H}{840}[41, 216, 27, 272, 27, 216, 41].$$

184. For example, let it be required to determine the area between the ordinates y_1 and y_2, in terms of the three equidistant ordinates y_1, y_2 and y_3, the common interval being h.

We must assume

$$y = A + Bx + Cx^2;$$

then taking the origin at the foot of y_1,

$$A = \int_0^h y\,dx = h\left[A + \frac{Bh}{2} + \frac{Ch^2}{3}\right],$$

from which A, B and C must be eliminated by means of the equations

$$y_1 = A,$$
$$y_2 = A + Bh + Ch^2,$$
$$y_3 = A + 2Bh + 4Ch^2.$$

Solving these equations, we obtain

$$A = y_1,$$
$$Bh = \frac{-3y_1 + 4y_2 - y_3}{2},$$
$$Ch^2 = \frac{y_1 - 2y_2 + y_3}{2};$$

If we make a slight modification in the ratios of these last coefficients by substituting for each the nearest multiple of 42, we have

$$A = \frac{H}{840}[42, 210, 42, 252, 42, 210, 42],$$

(the denominator remaining unchanged, since the sum of the coefficients is still 840), which reduces to

$$A = \frac{H}{20}[1, 5, 1, 6, 1, 5, 1].$$

This result is known as *Weddles' Rule* for six intervals. The value thus given to the mean ordinate is evidently a very close approximation to that resulting from Cotes' method, the difference being

$$\frac{1}{840}[y_1 + y_7 + 15(y_3 + y_5) - 6(y_2 + y_6) - 20y_4].$$

and substituting

$$A = \frac{h}{12}(5y_1 + 8y_2 - y_3).$$

185. It is, however, to be noticed, that when the ordinates are symmetrically situated with respect to the area, if n is *even*, the parabolic curve may be assumed of the $(n+1)$th degree. For example, in Art. 180, $n = 2$, but the curve was assumed of the third degree. Inasmuch as A, B, C and D cannot all be expressed in terms of y_1, y_2, and y_3, we see that a variety of parabolic curves of the third degree can be passed through the extremities of the measured ordinates, but all of these curves have the same area.*

Application to Solids.

186. If y denotes the area of the section of a solid perpendicular to the axis of x, the volume of the solid is $\int y\,dx$, and

* This circumstance indicates a probable advantage in making n an even number when repeated applications of the rules are made. Thus, in the case of six intervals, we can make three applications of Simpson's first rule, giving

$$A = \frac{H}{18}[1, 4, 2, 4, 2, 4, 1], \quad \ldots\ldots\ldots \quad (1)$$

or two of Simpson's second rule, giving

$$A = \frac{H}{16}[1, 3, 3, 2, 3, 3, 1]. \quad \ldots\ldots\ldots \quad (2)$$

In the first case, we assume the curve to consist of three arcs of the third degree, meeting at the extremities of the ordinates y_3 and y_5; but, since each of these arcs contains an undetermined constant, we can assume them to have common tangents at the points of meeting. We have therefore a *smooth*, though not a continuous curve. In the second case, we have two arcs of the third degree containing no arbitrary constants, and therefore making an angle at the extremity of y_4. It is probable, therefore, that the smooth curve of the first case will in most cases form a better approximation than the broken curve of the second case.

In confirmation of this conclusion, it will be noticed that the ratios of the coefficients in equation (1) are nearer to those of Cotes' coefficients for $n = 6$, given in the preceding foot-note, than are those in equation (2).

therefore the approximate rules deduced in the preceding articles apply to solids as well as to areas. Indeed, they may be applied to the approximate computation of any integral, by putting y equal to the coefficient of dx under the integral sign.

The areas of the sections may of course be computed by the approximate rules.

Woolley's Rule.

187. When the base of the solid is rectangular, and the ordinates of the sections necessary to the application of Simpson's first rule are measured, we may, instead of applying that rule, introduce the ordinates directly into the expression for the area in the following manner.

Taking the plane of the base for the plane of xy, and its centre for the origin, let the equation of the upper surface be assumed of the form

$$z = A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y + Ixy^2 + Jy^3.$$

Let $2h$ and $2k$ be the dimensions of the base, and denote the measured values of z as indicated in Fig. 35. The required volume is

$$V = \int_{-h}^{h} \int_{-k}^{k} z \, dy \, dx.$$

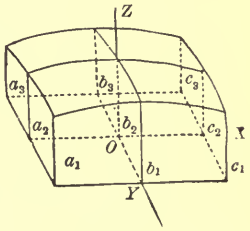

FIG. 35.

This double integral vanishes for every term containing an odd power of x or an odd power of y: hence

$$V = 4Ahk + \frac{4Dh^3k}{3} + \frac{4Fhk^3}{3},$$

$$= \frac{hk}{3}[12A + 4Dh^2 + 4Fk^2]. \quad \ldots\ldots (1)$$

By substituting the values of x and y in the equation of the surface, we readily obtain

$$b_2 = A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a_1 + a_3 + c_1 + c_3 = 4A + 4Dh^2 + 4Fk^2, \quad . \quad . \quad . \quad (3)$$

$$a_2 + c_2 + b_1 + b_3 = 4A + 2Dh^2 + 2Fk^2. \quad . \quad . \quad . \quad (4)$$

From these equations two very simple expressions for the volume may be derived; for, employing (2) and (4), equation (1) becomes

$$V = \frac{2hk}{3}(a_2 + b_1 + 2b_2 + b_3 + c_2); \quad . \quad . \quad . \quad . \quad (5)$$

and employing (2) and (3),

$$V = \frac{hk}{3}(a_1 + a_3 + 8b_2 + c_1 + c_3) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (5) is known as *Woolley's Rule;* the ordinates employed are those at the middles of the sides and at the centre; in (6), they are at the corners and at the centre.

Examples XV.

1. Apply Simpson's Rule to the sphere, the hemisphere, and the cone, and explain why the results are perfectly accurate.

2. Apply Simpson's Second Rule to the larger segment of a sphere made by a plane bisecting at right angles a radius of the sphere.

$$\frac{9\pi a^3}{8}.$$

3. Find by Simpson's Rule the volume of a segment of a sphere, b and c being the radii of the bases, and h the altitude.

$$\frac{\pi h}{6}(3b^2 + 3c^2 + h^2).$$

4. Find by Simpson's Rule the volume of the frustum of a cone, b and c being the radii of the bases, and h the altitude.

$$\frac{\pi h}{3}(b^2 + bc + c^2).$$

5. Compute by Simpson's First and Second Rules the value of $\int_0^1 \frac{dx}{1+x}$, the common interval being $\frac{1}{12}$ in each case.

The first rule gives 0.69314866; the second rule gives 0.69315046. The correct value is obviously $\log_e 2 = 0.6931471 8$.*

6. Find the volume considered in Art. 187, directly by Simpson's Rule, and show that the result is consistent with equations (5) and (6).

$$V = \frac{hk}{9}[a_1 + a_3 + c_1 + c_3 + 4(a_2 + b_1 + b_3 + c_2) + 16b_2].$$

7. Find, by elimination, from equations (5) and (6), Art. 187, a formula which can be used when the centre ordinate is unknown.

$$V = \frac{hk}{3}[4(a_2 + b_1 + b_3 + c_2) - (a_1 + a_3 + c_1 + c_3)].$$

* The error in the eighth place of decimals is therefore 148 by the First Rule, and 328 by the Second Rule, the First Rule giving the better result as anticipated in the foot-note of p. 220. Using the same ordinates, Weddles' Rule (see foot-note, p. 219) gives the extremely accurate result 0.69314722, the error being only 4 in the eighth place.

CHAPTER IV.

MEAN VALUES AND PROBABILITIES.

XVI.

The Average or Arithmetical Mean.

188. A MEAN of several values of a quantity is an intermediate value such that, when it is substituted for each of the given values in performing a certain operation, the result is unchanged. For example, if the result of the operation is the product of the given values, the mean value found is that known as the *geometrical mean*. But the usual and most simple mean value is that obtained when the operation is that of summation. This is known as the *average* or *arithmetical mean*.* Thus, if M denotes the average of n values of y denoted by $y_1, y_2, \ldots y_n$, we have

$$nM = y_1 + y_2 + \ldots + y_n. \quad . \quad . \quad . \quad . \quad (1)$$

Here the aggregate represented by either member of equation (1) is the same whether the n given values of y be taken or the mean value be taken n times.

* A mean value of any other kind can usually be defined by the aid of the arithmetical mean. Thus, the geometrical mean is the quantity whose logarithm is the mean of those of the given quantities; the harmonic mean is that whose reciprocal is the mean of the reciprocals. The *mean error* in the Theory of Least Squares is the error whose square is the mean of the squares of the errors.

189. When a number p_1 of the quantities has a common value y_1, a number p_2 has a common value y_2, and so on, the total number n of formula (1) is equal to $p_1 + p_2 + \ldots$ or Σp, and the formula becomes

$$\Sigma p \cdot M = p_1 y_1 + p_2 y_2 + \ldots = \Sigma p y. \quad . \quad . \quad (2)$$

The numbers $p_1, p_2, \ldots$ are called the *weights* of $y_1, y_2, \ldots$, and the mean is called the *weighted arithmetical mean* of the several values.

The *mean value* of a quantity which admits of a continuous series of values (which is the subject of the present Chapter) is a modification of the M of equations (1) and (2), in deriving which, integration takes the place of summation.

The Mean of a Continuous Variable.

190. Consider all the values of a variable which varies continuously between certain extreme values, and suppose a large number n of these values to be chosen and their mean taken. Then, supposing the manner of selection to be such that we can pass to the limit when n is indefinitely increased, we shall have a mean value which depends upon *all* the values of the variable in question, and is, therefore, properly called a mean value of the variable. But the value of this mean obviously depends also upon the method in which the n values were selected. For example, in finding the mean velocity of a point describing a straight line with variable velocity, we shall arrive at a certain result if we take the n values of the velocity at equal intervals of time; but the result will be different if we select the velocities with which the point passes equally distant points of its path.

191. When the variable in question is a function of some single independent variable, the mean obtained by taking *equi-*

distant values of the independent variable is called *the mean value of the function* for the range of values given to the independent variable. Let $y = f(x)$ be the function, represented, as in Fig. 8, p. 121, by the curve CD; then the figure illustrates the mode in which n values of the function or ordinate are taken in finding the mean value of the function for all values of x between $OA = a$ and $OB = b$. These values are the $y_1, y_2, \ldots y_n$ of Art. 99 erected at the common interval Δx, where $n\,\Delta x = b - a$.

Now, multiplying equation (1), Art. 188, by Δx, we have

$$n\,\Delta x \cdot M = y_1\,\Delta x + y_2\,\Delta x + \ldots + y_n\,\Delta x,$$

or

$$(b - a)M = \Sigma y \Delta x.$$

Here M is the mean of the n actual values of y, and the mean value required is found by passing to the limit when n is indefinitely increased.

It is shown in Art. 99 that the limit of $\Sigma y \Delta x$ is $\int_a^b y\,dx$, which is the area $CABD$ in Fig. 8. Hence the mean value of $f(x)$ for all values of x between a and b is given by the equation

$$(b - a)M = \int_a^b f(x)\,dx. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

192. The expression *mean ordinate* of any portion of a curve is always, unless otherwise stated, held to signify the mean ordinate *regarded as a function of the abscissa.* Hence it is the height of the rectangle of which the base is the projection of the curve on the axis of x, and the area is equal to that included between the curve, the base and the extreme ordinates. Thus, "the mean ordinate" of a semicircle whose radius is a is found by dividing the area, $\frac{1}{2}\pi a^2$, by the

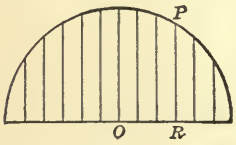

FIG. 36.

base $2a$; whence $M = \frac{1}{4}\pi a$. This is the average value of perpendiculars erected at equal distances along the diameter. See Fig. 36.

193. On the other hand, if the average value of perpendiculars let fall from equidistant points on the arc is required (see Fig. 37), it is necessary to express the perpendicular as a function of the arc or angle subtended at the centre. Denoting this (as measured from one extreme radius) by θ, the perpendicular is $a \sin\theta$, and the value of this mean is, by equation (3),

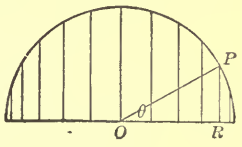

FIG. 37.

$$M = \frac{a\int_0^\pi \sin\theta\, d\theta}{\pi} = \frac{2a}{\pi}.$$

The Mean of Equally Probable Values.

194. The expression mean value of a variable quantity selected under given circumstances is often used to designate the mean of all the values which are *equally probable* under the circumstances. A point is said to be taken *at random* upon a line of given length when it is equally likely to fall upon any one of any equal segments of the line. Hence the first of the mean values of PR found in the preceding articles may be called the mean value when R is taken at random upon the diameter, and the second is the mean value when P is taken at random upon the semicircumference.

So also M in equation (1), Art. 188, is the mean value of y when $y_1, y_2, \ldots y_n$ are equally probable, and are the only possible values of y. In this case, the finite number n is *the total number of cases* which are possible and equally probable.

It will be convenient also in equation (3), Art. 191, to speak

of $b - a$ (which takes the place of n when it is increased indefinitely in passing to the limit) as *the total number of cases*, and of the definite integral in the second member as *the aggregate* of the y's in the total number of cases.

The Mean of a Function of Two Variables.

195. When the quantity whose mean is required depends upon two variables admitting of continuous values under certain restrictions, we may represent it by the z-coordinate of a surface of which the independent variables x and y are the other two rectangular coordinates. The restrictions imposed upon the values of x and y now limit the foot of the z-coordinate to a certain area in the plane of xy. If these restrictions consist of fixed limits between which x must lie, together with other fixed limits between which y must lie, this area is a rectangle. It may, however, be an area of any other shape; the limiting values of y will then be different for different values of x. In choosing the points for which the values of z are taken in forming the mean, the values of y, for a given value of x, are supposed to be taken at equal intervals Δy between the limits corresponding to that particular value of x. In like manner, the values of x chosen are taken at equal intervals Δx, so that the points (x, y) are *uniformly distributed* over the area. Thus, if the area contains a large number, n, of elementary rectangles of dimensions Δy and Δx, then one value of z is taken corresponding to each element, $\Delta y\, \Delta x$, of area.

196. Now, putting z for y in equation (1), Art. 188, and multiplying each member by $\Delta y\, \Delta x$, we have

$$n\, \Delta y\, \Delta x \cdot M = \Sigma z \Delta y \Delta x. \quad . \quad . \quad . \quad . \quad (1)$$

On passing to the limit when Δx and Δy are indefinitely

decreased, M becomes the mean value required. The limiting value of $n\Delta y\Delta x$ is the area mentioned above; and its value is

$$A = \iint dy\, dx,$$

where the limits of integration are given in the form of restrictions upon the values of x and y. The limiting value of the second member of equation (1) is the double integral

$$\iint z\, dy\, dx,$$

taken with the same limits; or, as expressed in Art. 126, *integrated over the area A;* and its value, as shown in that article, is the volume of the cylindrical solid whose upper surface is in the surface $z = \phi(x, y)$, and whose base is the area A. Thus we have the equation

$$M \cdot \iint dy\, dx = \iint \phi(x, y) dy\, dx, \quad . \quad . \quad . \quad (2)$$

which defines the mean value of a function of two variables, and shows that, when thus geometrically represented, it is the height of a cylinder having the area of integration as base and a volume equal to that of the solid described above.

197. The "mean ordinate" for a given surface referred to rectangular coordinate planes is always understood to signify the mean value of z, thus considered as a function of x and y. If the volume and the area of the base are known, the mean is at once known. For example, the mean ordinate of a hemispherical surface whose radius is a (of which Fig. 36 may represent a central section) is given by

$$\pi a^2 \cdot M = \tfrac{2}{3}\pi a^3, \qquad \text{whence} \qquad M = \tfrac{2}{3}a.$$

When regarded as the mean of equally probable values, as in Art. 194, it is implied that every admissible pair of values of x and y has the same probability. That is to say, in the geometrical illustration, every position of the point R within the area is equally probable, and this is expressed by saying that the point R falls *at random* upon the base.

It will be convenient here also (compare Art. 194) to speak of the area, which in equation (2) of Art. 196 takes the place of n, as *the total number of equally probable cases;* and of the second member of the equation as *the aggregate* of the function z in the total number of cases.

The Mean of Values not Equally Probable.

198. When the variable in question is expressed as a function of a single variable, it may happen that the mean required is not that which we have defined as *the mean value of the function* between certain limits. For example, if the average height above the base of a point on the surface of the hemisphere is required, it may be expressed by $a \sin\theta$, as in Fig. 37 (which we now take to represent a central section perpendicular to the circular base). But the mean now required is *not* the mean value of PR considered as a function of θ (which was found in Art. 193); for all values of θ are not now equally probable. In the present problem, the "total number of cases" is represented by the area of the convex surface of the hemisphere upon which P is said to fall at random. Instead of a single case, or an equal number of cases, corresponding to each element of arc $ad\theta$, we have a number proportional to the area of the element of surface dS which corresponds to that element of arc, and the surface elements corresponding to different elements of arc are unequal areas. Thus we have a case of the weighted mean of Art. 189; in which the weight p is represented by the element of surface, and Σp by the

whole surface. The length of the element dS is the circumference whose radius is $a \cos\theta$; hence

$$dS = 2\pi a^2 \cos\theta \, d\theta,$$

and the surface of the hemisphere is $2\pi a^2$. Thus M is determined by

$$2\pi a^2 \cdot M = \int a \sin\theta \, dS = 2\pi a^3 \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta = \pi a^3,$$

which gives $M = \frac{1}{2}a$ for the average height of a point on the surface of the hemisphere.

199. In the foregoing problem, if P be joined to the centre O, θ is the elevation of the direction OP above the horizontal plane (that of the base of the hemisphere). OP is said to have a *random direction in space* because P was taken at random upon the spherical surface. Hence the problem solved above is the same as that of finding the mean initial *vertical* velocity of a body projected upward in a random direction with the velocity a.

Again, the horizontal velocity of such a body with initial velocity V is $V\cos\theta$. Hence the mean horizontal velocity is found by

$$2\pi V^2 \cdot M = 2\pi V^3 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{\pi^2 V^3}{2};$$

whence $M = \frac{1}{4}\pi V$.

200. Even when we use the ultimate element of surface upon which the variable point falls at random (which may be denoted by d^2S), and perform a double integration, the area of the element may be regarded, as in Art. 198, as giving the weight to be attributed to that special value of the function which corresponds to the element. It is only in the case of the plane element $dy\,dx$ that the area of an element is constant, so as to give equal weights to values selected one from each element.

For example, in using polar coordinates, let the area in Fig. 22, p. 172, represent the limits of integration, and let *one* value of the quantity whose mean is required be selected for each of the small areas represented. Then, it is only by using the varying value of the area of an element, namely $r\,dr\,d\theta$, that we give proper weight to each value *to represent uniform distribution over the area.* If we use $dr\,d\theta$ simply, we shall find quite a different mean value, having thus given more weight comparatively to values of the function corresponding to small values of r than should be done to represent uniform distribution.

The Centre of Position of n Points.

201. The mean distances of n points in a plane from fixed straight lines in the plane, and more generally of n points in space from fixed planes, form an important application of the principles of mean values.

In the case of points in a plane, let their positions be referred to rectangular axes, of which that of y is the line from which the mean distance is required. Then, if the points are (x_1, y_1), (x_2, y_2), ... (x_n, y_n), and $\bar{x}$ denotes the mean distance from the axis of y, its value is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) = \frac{1}{n}\Sigma x. \quad . \quad . \quad (1)$$

In like manner, the mean distance of the points from the axis of x is

$$\bar{y} = \frac{1}{n}(y_1 + y_2 + \ldots + y_n) = \frac{1}{n}\Sigma y. \quad . \quad . \quad (2)$$

202. We shall now show that the point which has these mean values for its coordinates has the property that its dis-

tance from *any* straight line in the plane is the mean of the distances of the several points from that line.

In the first place, let the line in question be parallel to an axis, say that of y, and at a distance h from it. Then (taking h as positive when the line is on the left of the axis), the distance of (x_1, y_1) from it is $x_1 + h$. In like manner, the distance of (x_2, y_2) is $x_2 + h$, and so on. Hence the mean distance is $\frac{1}{n}\Sigma(x + nh) = \frac{1}{n}\Sigma x + h = \bar{x} + h$, by equation (1). But this is the distance of the point $(\bar{x}, \bar{y})$ from the line in question.

Next let the line be oblique to the axes. It is shown in Analytical Geometry that the perpendicular upon a line from (x, y) is of the form

$$p = Ax + By + C,$$

where A, B, and C are certain constants; in other words, p is a linear function of x and y. Hence we have, for the distances of (x_1, y_1), (x_2, y_2), etc., from the line

$$\begin{aligned} p_1 &= Ax_1 + By_1 + C, \\ p_2 &= Ax_2 + By_2 + C, \\ &\cdots\cdots\cdots \\ p_n &= Ax_n + By_n + C; \end{aligned}$$

and adding,

$$\Sigma p = A \cdot \Sigma x + B \cdot \Sigma y + nC.$$

Dividing by n, the mean distance is, by equations (1) and (2),

$$\bar{p} = A\bar{x} + By + C;$$

but this is the distance of the point $(\bar{x}, \bar{y})$ from the given oblique line.

The point $(\bar{x}, \bar{y})$ whose position thus determines the mean distance of the given points from any straight line in the plane is called their *Centre of Position*.

203. In like manner, when n points $P_1, P_2, \ldots P_n$ in space are referred to rectangular planes, their mean distances from these planes are

$$\bar{x} = \frac{1}{n}\Sigma x, \qquad \bar{y} = \frac{1}{n}\Sigma y, \qquad \bar{z} = \frac{1}{n}\Sigma z;$$

and it can be shown that the mean distance of the points from any plane is the distance of the point $(\bar{x}, \bar{y}, \bar{z})$ from that plane. The point which thus determines the mean distance from all planes is called the Centre of Position of the points. It is also called, from its occurrence in Statics, the *Centre of Gravity* of n equal particles situated at the points $P_1, P_2, \ldots P_n$.

204. If a number p_1 of the equal particles coincide at the point P_1, p_2 of them at P_2, and so on, the centre of gravity of the particles is at a distance from any given plane which is the weighted mean (Art. 189) of the distances of the point P_1, P_2, etc. from the plane. Thus, referring to rectangular coordinate planes, its distance $\bar{x}$ from that of yz is given by the equation

$$\bar{x} \cdot \Sigma p = p_1 x_1 + p_2 x_2 + \ldots = \Sigma p x.$$

Multiplying both sides by the mass of one of the equal particles, the equation becomes

$$\bar{x} \cdot \Sigma m = m_1 x_1 + m_2 x_2 + \ldots = \Sigma m x, \quad . \quad . \quad (1)$$

where m_1, m_2, etc. are the masses of *unequal* particles situated at P_1, P_2, etc., and Σm is the total mass.

The point $(\bar{x}, \bar{y}, \bar{z})$ whose three coordinates are similarly defined is *the centre of gravity of unequal particles*.

The second member of equation (1), that is, the aggregate of the distances multiplied by the masses, is called in Mechanics the *Statical Moment* of the total mass Σm with respect to the plane of yz.

The Centre of Gravity of a Continuous Body.

205. The property of the centre of gravity given in the preceding articles obviously extends to continuous bodies. That is to say, the position of this point, which is sometimes called the *Centroid*, is determined by the mean distances of the particles from three given planes; and, when given, it determines the mean distance from *any* plane. If the body is *homogeneous*, that is to say, if the same quantity of matter is contained in equal elements of volume, the mean distances are the ordinary arithmetical means. On the other hand, if the body is not homogeneous, its density at a given point constitutes the weight to be attributed to the element at that point, supposing, as in Art. 135, the geometric elements to be all equal. In any case, the variable density is simply used as a factor of the element both in finding the total mass (as in Art. 135) and in finding the statical moment.

206. The centre of gravity of a homogeneous solid is also called *the centre of gravity of the volume*, and the integral of xdV (the factor of density being omitted) is called the statical moment of the volume. So also the centre of gravity of a thin homogeneous plate of uniform thickness is called *the centre of gravity of the area*, and the integral of xdA is called the statical moment of the area. Thus, for the circle $x^2 + y^2 = a^2$, the statical moment with respect to the axis of y of the semicircle on its right is

$$\int x\,dA = 2\int_0^a xy\,dx = 2\int_0^a \sqrt{(a^2-x^2)}\,x\,dx = -\tfrac{2}{3}(a^2-x^2)^{\frac{3}{2}}\Big]_0^a = \tfrac{2}{3}a^3.$$

Dividing by the area $\frac{1}{2}\pi a^2$, we have, for the abscissa of the centre of gravity of the semicircle,

$$\bar{x} = \frac{4a}{3\pi}.$$

207. One or more of the coordinates of the centroid may be obvious from considerations of symmetry. For example, the centroid of a plane area or of a plane curve is in the plane; that of a straight line is at its middle point; that of a sphere or of an ellipsoid is at its centre; that of a hemisphere is on its central radius; that of a right cone is on its axis.

Again, separation of the body into elements whose centroids are known may lead to similar results. For example, an oblique cone can be separated into circular elements of uniform thickness whose centroids are therefore at their centres. These points are all situated on the geometrical axis, therefore the centroid of the cone is also on that axis. In like manner, the centre of gravity of any triangle is upon a medial line. It follows that it is at the intersection of the three medial lines. It is obvious also that, if three equal particles be situated at the vertices of any triangle, their centre of gravity will be at the same point. Hence the distance of the centre of gravity of a triangle from any straight line in the plane is the arithmetical mean of the distances of the vertices.

208. When integration is required, the element of moment employed may be the moment of any convenient element of the mass or volume. For example, in the case of a hemisphere of radius a, it is necessary to use integration in finding the distance from the base. We may take for element of volume the hemispherical shell of radius r and thickness dr, of which we have already found in Art. 198 the mean distance from the base to be one-half of the radius. Therefore, since

the volume of the shell is $2\pi r^2\,dr$, its moment is $\pi r^3\,dr$. Integrating, we have for the hemisphere

$$\text{moment} = \pi \int_0^a r^3\,dr = \frac{\pi a^4}{4}.$$

Dividing by the volume, which is $\frac{2}{3}\pi a^3$, we find the height of the centre of gravity above the base to be $\frac{3}{8}a$. This is therefore the average distance from the base of *all* the points within the hemisphere.

The moment might be found as the result of a simple integration, in this case, also by using the circular element of volume parallel to the base. The method employed above is particularly adapted to a sphere of varying density if the density is a function of the distance from the centre only.

Average Squared Distance of Points from a Plane.

209. The mean square of the distances of points in a plane area from a straight line in the area, and that of the distances of points in a volume from a fixed plane, or from a fixed straight line, have important application in Mechanics.

In the case of the area A, if we denote the mean squared distance from the axis of y by $\overline{x^2}$, it is found from the formula

$$A \cdot \overline{x^2} = \iint x^2\,dy\,dx,$$

where the integration extends over the area A. The second member of this equation is known as *the moment of inertia* of the area with respect to the axis of y, and is usually denoted by I. The distance k, such that $k^2 = \overline{x^2}$, is called *the radius of gyration* of the area with respect to the same axis; thus the equation becomes $k^2 A = I$.

210. As in finding the statical moment, so in finding the moment of inertia, it may be convenient to use some other

element of area in place of $dy\,dx$. For example, in finding the moment of inertia, with respect to the axis of y, of the loop of the lemniscate $r^2 = a^2 \cos 2\theta$, of which the area is found in Art. 109, we use the polar element of area $r dr\, d\theta$. The distance of the element from the axis is $x = r \cos\theta$. Hence

$$I = 2\int_0^{\frac{\pi}{4}}\int_0^r r^3 \cos^2\theta\, dr\, d\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{4}} \cos^2 2\theta \cos^2\theta\, d\theta.$$

Putting $\phi = 2\theta$ (see Art. 96), this becomes

$$I = \frac{a^4}{8}\int_0^{\frac{\pi}{2}} \cos^2\phi(1 + \cos\phi)d\phi = \frac{a^4}{8}\left[\frac{\pi}{4} + \frac{2}{3}\right];$$

and since, as found in Art. 109, $A = \frac{1}{2}a^2$, $k^2 A = I$ gives

$$k^2 = \frac{3\pi + 8}{48}a^2.$$

211. In the case of a solid the "moment of inertia" and "radius of gyration" depend upon the distances of the particles, not from a plane, but from a given straight line in space. Thus, referring the body to rectangular coordinate planes, if r denotes the perpendicular upon the axis of z from the point P at which the particle m is situated, the moment of inertia *for the axis of* z is

$$I_z = \Sigma m r^2 = M k_z^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now since $r^2 = x^2 + y^2$,

$$\Sigma m r^2 = \Sigma m x^2 + \Sigma m y^2. \quad . \quad . \quad . \quad . \quad (2)$$

Thus the moment of inertia is the sum of two such integrals as are defined in the preceding article. Dividing equation (2) by M, the total mass, it becomes

$$k_z^2 = \overline{x}^2 + \overline{y}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

212. The value of I can often be found directly from its definition Σmr^2 instead of using the second member of equation (2). For example, to find the moment of inertia of a homogeneous cylinder of radius a and height h with reference to its geometrical axis. The integral expression for the entire moment of inertia of a volume is

$$I = \int r^2 \, dV,$$

where dV is an element such that r is the distance of every particle in it from the line or axis of reference. In other words, dV is the cylindrical element or differential of volume, which in Art. 120 was seen to be particularly adapted to finding the volume of solids of revolution. In the present case, the height of the cylindrical surface is h and its circumference is $2\pi r$. Thus $dV = 2\pi hr \, dr$, and $dI = 2\pi hr^3 \, dr$; whence

$$I = 2\pi h \int_0^a r^3 \, dr = \tfrac{1}{2}\pi h a^4,$$

and, since $V = \pi h a^2$, the corresponding value of k^2 is $\frac{1}{2}a^2$.

213. As we should expect, this value of k^2 is independent of h; it therefore applies to a circular lamina; in other words, it is the value of k^2 *for a circle with reference to its axis*, that is to say, a line passing through its centre and perpendicular to its plane.

The symmetry of the cylinder shows that, referring its base to rectangular diameters as axes of x and y, Σmx^2 and Σmy^2

must be equal. Therefore, $\overline{x^2} = \overline{y^2}$; hence from equation (3), Art. 211, $\overline{x^2}$ is one-half the value of k^2 found above, that is to say $\frac{1}{4}a^2$.

This is the value of k^2 for a circular area with reference to a diameter; but it must be noticed that it is *not* a value of k^2 for the cylinder. For that figure, the value of k_y must be determined by

$$Mk_y^2 = I_y = \Sigma mx^2 + \Sigma mz^2,$$

in which z is a coordinate in the direction of the geometrical axis. See Ex. 23 below.

More extended discussions of these integrals and the relation between them for different axes will be found in Treatises on Mechanics.

Examples XVI.

1. Find the mean value of the tangent of an arc between 0 and 45°. $\frac{2\log_e 2}{\pi} = .441.$

2. Find the mean value of $\tan^{-1} x$ when $0 < x < 1$.

$$\frac{\pi}{4} - \frac{1}{2}\log 2 = .439.$$

3. A number n is divided at random into two parts. Find the mean value of the product. $\frac{1}{6}n^2$.

4. Find the mean value of the square of one of two parts of the line a taken at random. $\frac{1}{3}a^2$.

5. Find the mean value of the square root of one of two random parts of a. $\frac{2}{3}\sqrt{a}$.

6. Find the mean value of the chord drawn from a point on the circumference of the circle whose radius is a, all directions of the chord being equally probable. $\frac{4a}{\pi}$.

7. What is the average distance of points within a sphere, radius a, from the centre? $\frac{3}{4}a$.

8. The half of a right cone of height h and radius of base b stands on its triangular surface as base ; what is the mean ordinate of the curved surface ? $\frac{\pi b}{6}$.

9. What is the mean height above the base of a point in the surface given in Ex. 8 ? $\frac{4b}{3\pi}$.

10. Find the mean value of the chord AB of a sphere, radius a, drawn from a fixed point A on the surface, (α) when all positions of B on the sphere are equally probable ; (β) when all directions (see Art. 199) of AB are equally probable. (α) $\frac{4}{3}a$; (β) a.

11. Show that, when a space is described by a point with variable velocity, the mean velocity at a random instant is the ordinary "average velocity," or whole space divided by the whole time. Show also that for a body falling from rest (where $v = gt$, $v^2 = 2gs$) this mean velocity is $\frac{1}{2}$ of the final velocity, while the mean velocity of passing a random point is $\frac{2}{3}$ of the final velocity.

12. Regarding the earth as a sphere, what is the average latitude of a point in the northern hemisphere ? $32^\circ\ 41'\ 15''$.

13. Given that the horizontal range of a projectile fired with the velocity V and the angle of elevation θ is $R = \frac{V^2}{g} \sin 2\theta$, show that the mean range of a projectile fired in a random direction is two-thirds of the maximum range.

14. Find the centre of gravity of the area enclosed between the parabola $y^2 = 4mx$ and the double ordinate corresponding to the abscissa a. $\bar{x} = \frac{3}{5}a$.

15 Find the centre of gravity of the area between the semicubical parabola $ay^2 = x^3$ and the double ordinate which corresponds to the abscissa a. $\bar{x} = \frac{5}{7}a$.

16. Find the coordinates of the centre of gravity of the area between the axes and the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$

$\bar{x} = \frac{1}{5}a$; $\bar{y} = \frac{1}{5}b$.

17. Find the centre of gravity of the area between the *cissoid* $y^2(a-x) = x^3$ and its asymptote. $\bar{x} = \frac{5}{6}a$.

18. Find the centre of gravity of the area enclosed between the positive directions of the coordinate axes and the four-cusped hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad \bar{x} = \bar{y} = \frac{256a}{315\pi}.$$

19. Given the cycloid,

$$y = a(1 - \cos\psi), \qquad x = a(\psi - \sin\psi),$$

find the distance of the centre of gravity of the area from the base.

$$\bar{y} = \frac{5a}{6}.$$

20. Determine the distance from the base of the centre of gravity of a hemisphere when the density varies inversely as the distance from the centre. $\frac{1}{3}a$.

21. Find the mean squared distance from the base for the particles of a homogeneous hemisphere. $\bar{x^2} = \frac{1}{5}a^2$.

22. Determine the radius of gyration of a sphere for a diameter by means of the result of Ex. 21, and also directly by the method of Art. 212. $k^2 = \frac{2}{5}a^2$.

23. Determine the radius of gyration of a cylinder with respect to a diameter of the base. $k^2 = \frac{1}{4}a^2 + \frac{1}{3}h^2$.

24. Find the radius of gyration of the area between a parabola and a double ordinate $2y$ perpendicular to its axis, with respect to a perpendicular to its plane passing through its vertex.

$$k^2 = \frac{3}{7}x^2 + \frac{1}{5}y^2.$$

25. Determine the radius of gyration of a paraboloid about its axis, b denoting the radius of the base. $k^2 = \frac{1}{3}b^2$.

26. The cardioid

$$r = a\ (1 - \cos\theta)$$

revolves about the initial line; determine the radius of gyration of the solid formed with respect to this line. $k^2 = \frac{24}{35}a^2$.

XVII.

Mean Distances from a Fixed Point.

214. We have seen that the mean value of a variable depends not only upon the restrictions imposed in the question upon the values of the variable, but upon the distribution of the admissible values. This distribution is analytically expressed in the selection of the element employed, and the restrictions are expressed by the limits of integration. For example, in finding the mean distance, from a fixed point O, of the points on a straight line AB of limited length, the variable is OP, Fig. 38, and the distribution of the values (expressed in words by the statement that P falls at random between the limits assigned), is provided for by selecting the element dy, where y is the distance of P from a fixed point of the line. Taking for this point A the foot of the perpendicular $OA = a$ from the point O, the expression for the variable in question is $OP = \sqrt{(a^2 + y^2)}$. Thus, if A and B are given as the limiting positions of P, the "whole number of cases" is represented by $AB = b$, which is the integral of the element dy between the limits 0 and b. The mean distance is then given by the equation

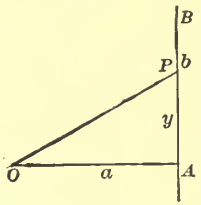

FIG. 38.

$$bM = \int_0^b \sqrt{(a^2+y^2)}\, dy = \frac{1}{2}\left[b\sqrt{(a^2+b^2)} + a^2 \log \frac{b+\sqrt{(a^2+b^2)}}{a} \right];$$

see formula (L), p. 125.

215. As a particular case of this result, putting $b = a$, we have for the mean distance from one corner of a square of a point taken at random upon either of the opposite sides

$$M = \tfrac{1}{2}a[\sqrt{2} + \log(1 + \sqrt{2})] = 1.148a,$$

where a is a side of the square. It readily follows that this is the mean distance from the centre of a point on the perimeter of a square of side $2a$, and therefore $0.574a$ is the same mean when the side is a.

Again, putting $b = a\sqrt{3}$, so that the angle AOB is 60°, we find, for the mean distance from the centre of a point on the perimeter of an equilateral triangle circumscribing the circle whose radius is a,

$$M = a[1 + \tfrac{1}{6}\sqrt{3}\log(\sqrt{3} + 2)];$$

or, if s is a side of the triangle,

$$M = \tfrac{1}{12}s[2\sqrt{3} + \log(\sqrt{3} + 2)] = 0.398s.$$

216. If P in Fig. 38 is taken at random within the rectangle with vertex at O and sides a and b, the mean distance is the value of M in the equation

$$abM = \int_0^a \int_0^b \sqrt{(x^2 + y^2)}\,dy\,dx, \quad . \quad . \quad . \quad . \quad (1)$$

in which the "number of cases" is represented by the area of integration ab. Here, the result of the first integration would be

$$\frac{1}{2}\left[b\sqrt{(x^2 + b^2)} + x^2 \log \frac{b + \sqrt{(x^2 + b^2)}}{x}\right],$$

which gives a troublesome form to the second integration.

But, if we replace the upper limit b by mx, where m is constant, this becomes

$$\tfrac{1}{2}\{m\sqrt{(1 + m^2)} + \log[m + \sqrt{(1 + m^2)}]\}x^2 = Kx^2, \quad . \quad (2)$$

and the second integration becomes very simple. In doing this, we integrate over a triangle instead of a rectangle. Thus if, in Fig. 38, $b = ma$, the mean distance from O of a point within the triangle OAB is given by

$$\frac{1}{2}ma^2M_1 = \int_0^a \int_0^{mx} \sqrt{(x^2+y^2)}\,dy\,dx = \int_0^a Kx^2dx = \frac{Ka^3}{3}; \quad (3)$$

whence $M_1 = \dfrac{2Ka}{3m}$, where K is the constant defined in equation (2).

It is obvious that the value of the integral in equation (1) can thus be obtained by dividing the field of integration into two parts. Again, because the two parts of the rectangle ab are equal, if M_2 is the value of the mean in the second part, we shall have $M = \frac{1}{2}(M_1 + M_2)$.

217. The element in the double integral of the area, which here represents the number of cases, is $dy\,dx$, which we may denote by d^2A. In the single integral, the element is the result of integrating this for y; that is $dA = y\,dx$ taken between certain limits for y. Thus, in integrating over the half square OAC in Fig. 39, it is $dA = x\,dx$. Now, if M_0 denotes the mean value of the variable in question for this element, the result of the first integration of the expression for the aggregate must equal M_0dA; hence, M being the final mean, we have

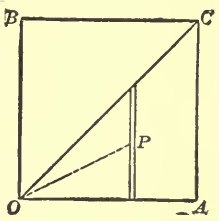

FIG. 39.

$$M \cdot A = \int M_0 dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

M_0 is here a function of the independent variable x, and M_0dA may be called the aggregate of the new cases introduced when x is changed to $x + dx$.

This equation is frequently useful in extending results already found. Thus, in Fig. 39, M_o is the mean value of OP for a side opposite to O of the square whose side is x; hence by Art. 215, $M_o = 1.148x$. Therefore, by equation (1),

$$\tfrac{1}{2}a^2M = 1.148\int_o^a x^2\,dx;$$

whence

$$M = \tfrac{2}{3} \times 1.148a = .765a.$$

Mean Distances between Two Variable Points.

218. When each of two points is taken at random upon a fixed line, the distance between them is a function of two variables, since the position of each point is, in that case, determined by a single variable. For example, if each of the points lies upon the circumference of the same circle of radius a, it is determined by an angle at the centre measured from some fixed radius OA. The problem is, in this case, to find the mean value of a chord under certain restrictions. If there are no further restrictions, so that each end of the chord is equally likely to fall at any point of the circumference, it is obvious by symmetry that we may assume one end to be at the fixed point A, and the other, B, to fall at random upon the semicircumference ABD in Fig. 40. This restricts the number of cases to π, and we have $AB = 2a \sin \frac{1}{2}\theta$. Therefore

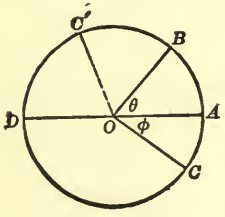

Fig. 40.

$$M = \frac{2a}{\pi}\int_o^\pi \sin\tfrac{1}{2}\theta\,d\theta = \frac{4a}{\pi}$$

is the mean value of a chord when all positions of the extremities are equally probable.

219. Now suppose that we require the mean value M_1 of the chord BC which cuts a fixed diameter AD. Then the two variables $\theta = AOB$ and $\phi = AOC$ in Fig. 40 determine the positions of the extremities, and each is restricted to values between 0 and π. The number of cases is now π^2, and the value of BC is $2a \sin \frac{1}{2}(\theta + \phi)$. Therefore

$$\pi^2 M_1 = 2a \int_0^\pi \int_0^\pi \sin \tfrac{1}{2}(\theta + \phi) d\theta \, d\phi = -4a \int_0^\pi \cos \tfrac{1}{2}(\theta + \phi) \Big]_{\theta=0}^{\theta=\pi} d\phi$$

$$= 4a \int_0^\pi (\cos \tfrac{1}{2}\phi + \sin \tfrac{1}{2}\phi) \, d\phi = 8a \Big[\sin \tfrac{1}{2}\phi - \cos \tfrac{1}{2}\phi \Big]_0^\pi = 16a;$$

whence $M_1 = \dfrac{16a}{\pi^2}$.

220. When the lines on which the random points fall have a common part (in which case zero will be a possible value of the distance), care must be taken that the expression for the distance does not become negative. For example, if both extremities of the chord B and C' fall at random in the upper semicircumference in Fig. 40, the expression for the chord is $BC' = 2a \sin \frac{1}{2}(\phi - \theta)$, which has a negative value when $\phi < \theta$. The aggregate formed by integrating over the range 0 to π for both variables would obviously be zero in this case. What we require is the mean of the *positive* value of this expression. This limits the number of cases to $\frac{1}{2}\pi^2$, namely those in which

$$\pi > \phi > \theta > 0;$$

thus, if we integrate first for θ, its limits are 0 and ϕ. Denoting the mean in this case by M_2, we have for the mean value of the chord which does not cut a given diameter

$$\tfrac{1}{2}\pi^2 M_2 = 2a\int_0^{\pi}\int_0^{\phi} \sin\tfrac{1}{2}(\phi-\theta)\,d\theta\,d\phi$$

$$= 4a\int_0^{\pi}(1-\cos\tfrac{1}{2}\phi)\,d\phi = 4a(\pi-2).$$

Therefore $$M_2 = \frac{8(\pi-2)a}{\pi^2}.$$

The whole number of chords which do not cut the diameter through A, inclusive of those which lie below the diameter, is π^2, the same as that of the chords which do cut the diameter. Therefore the mean value, when all restrictions are removed, is $M = \frac{1}{2}(M_1 + M_2)$. Accordingly, the values found in this and the preceding article are verified by the value of M found in Art. 218.

Mean Distances Connected with a Sphere.

221. In finding the mean distance of the point B taken at random within the volume of the sphere whose radius is a from the point A taken at random upon the surface, we may obviously take A as fixed. Taking this point as the pole in Fig. 41 (which represents a section through A, B and the centre), the distance is the radius vector of the point B, and the number of cases is the volume, $\frac{4}{3}\pi a^3$, of the sphere. In finding the aggregate of the r's, we may avoid triple integration by taking for dV, as in Art. 146, the volume generated by the element of area $r\,dr\,d\theta$ when the figure revolves about the initial line. Thus

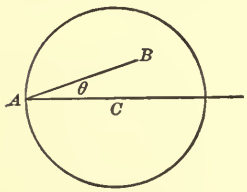

FIG. 41.

$$dV = 2\pi r^2 \sin\theta\,dr\,d\theta;$$

then the mean value of r is determined by

$$\tfrac{4}{3}\pi a^3 M = 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^3\, dr \sin\theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (2a\cos\theta)^4 \sin\theta\, d\theta = \tfrac{8}{5}\pi a^4.$$

Therefore $M = \frac{6}{5}a$.

222. This result may be extended to the case of two points both taken at random within the sphere by the method of Art. 217, which applies equally well when the total number of cases is not represented by an area. For, suppose N, the total number of cases to be expressed in terms of a single independent variable, say r, so that dN is the number of new cases introduced when r is changed to $r + dr$. Then, if M_0 is the mean value for these new cases, the final mean M will be determined by

$$M \cdot N = \int M_0 dN.$$

In the present problem, when both points are taken at random within the sphere whose radius is r, the whole number of cases is $N = (\frac{4}{3}\pi r^3)^2 = \frac{16}{9}\pi^2 r^6$, whence $dN = \frac{16}{9}\pi^2 . 6r^5 dr$. The "new cases," of which this is the number, are plainly those in which one of the points is on the surface of the sphere and the other anywhere within; hence, by the preceding article, $M_0 = \frac{6}{5}r$. Therefore

$$\tfrac{16}{9}\pi^2 a^6 M = \tfrac{16}{9}\pi^2 . 6 . \tfrac{6}{5} \int_0^a r^6\, dr;$$

whence $M = \frac{36}{35}a$.*

* It may be noticed that in this process if N when expressed in terms of the single variable is of the form kz^n and M_0 is of the form cz^m, the differentiation of

Random Parts of a Line or Number.

223. The division of a line (which may be taken to represent a number regarded as a continuous quantity capable of indefinite subdivision) into three parts at random involves the random fall of two points of division. Each point of division is equally likely to fall at any position on the line. Let AB be the line, and R and S the points of division; then, putting $AR = x$ and $AS = y$, it is to be noticed that x is *not* one of the three parts unless $y > x$. If this restriction is made, the parts are

$$x, \quad y - x, \quad a - y.$$

Without the restrictions, the mean value of x would of course be $\frac{1}{2}a$, and the mean value of $y - x$ would be zero, since for every case in which $y - x$ has a positive value there is a case in which it has an equal negative value. But, when the restriction is made, the mean value of $y - x$ is the same as the mean distance of two points taken at random on a straight line, in finding which it is necessary to exclude negative values, as in Art. 220.

224. That there is no distinction in kind between the three parts thus symbolized is clearly seen if we imagine the line bent into the circumference of a circle upon which three points fall at random. No change is made by taking one of the points as fixed.

It follows that the mean values of the three parts are equal. Again, since the sum of the corresponding terms which enter

---

N introduces the factor n, and the integration introduces the divisor $m + n$; so that the relation between M and M_0 for the same final value of z is

$$M = \frac{n}{m+n} M_0.$$

Thus, in Art. 217, we ha_ $M = \frac{2}{3} M_0$; in the present case, we have $M = \frac{2}{4} M_0$.

the three aggregates in each "case" (or mode of division) is a, the sum of the aggregates is a multiplied by the whole number of cases. Therefore the sum of the means must be a, and each mean is $\frac{1}{3}a$.

225. It is sometimes useful in problems involving two independent variables, like the present, to represent them by the coordinates of a point, and to consider the area of integration as in Art. 128. Thus, in Fig. 42, let the x of Art. 223 be the abscissa, OR, and y the ordinate of a point P referred to the rectangular axes OA and OB. Then the condition $y > x$, while each is positive and less than a, restricts P to the triangle OBC. It is now obvious that the mean value of x is the same as $\bar{x}$, the abscissa of the centre of gravity of the triangle OBC, which is $\frac{1}{3}a$, agreeing with the conclusion in the preceding article.

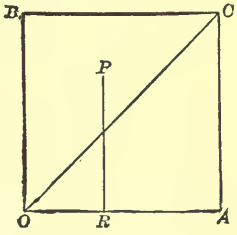

FIG. 42.

Again, the mean value M of x^2 is given by

$$\tfrac{1}{2}a^2M = \int_0^a \int_0^y x^2\,dx\,dy = \tfrac{1}{3}\int_0^a y^3\,dy = \tfrac{1}{12}a^4;$$

whence $M = \frac{1}{6}a^2$. This is in fact the square of the radius of gyration (Art. 209) of the area of integration OBC with reference to the axis of y.

226. If we make a real distinction between the parts, for instance, if we let x represent the *smallest* of the three parts, the area of integration is still further restricted. Thus, if the three parts in Art. 223 are in order of magnitude, $a - y$ being the greatest, the conditions are completely expressed by

$$0 < x < y - x < a - y.$$

Here, the first inequality shows that P is on the right of the line

$x = 0$, in Fig. 43, or OB. The second shows that P is on the left of the line $x = y - x$, or $y = 2x$, which is the line OD, joining O to D the middle point of BC. The third inequality shows that P is below the line $2y = a + x$, which is the line CE, where E is the middle point of OB in the figure. Thus P is restricted to the triangle OEF. The intersection F of the two lines is the point $(\frac{1}{3}a, \frac{2}{3}a)$. The mean value of x is therefore the abscissa of the centre of gravity of this triangle, namely $\frac{1}{9}a$. The mean value of y is the ordinate of the same point, which by Art. 207 is the mean of the ordinates of the vertices, that is $\frac{1}{3}(\frac{1}{2}a + \frac{2}{3}a) = \frac{7}{18}a$. Hence we have, for the mean values of the least, middle and greatest parts,

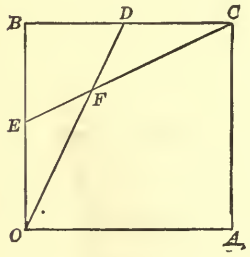

FIG. 43.

$$\frac{1}{9}a, \quad \frac{5}{18}a \quad \text{and} \quad \frac{11}{18}a.$$

Random Division into n Parts.

227. If a number a is divided into n parts at random, $n - 1$ of these parts may be taken as independent variables x, y, z etc., subject to the condition that each shall have a positive value and that their sum shall not exceed a; thus,

$$x + y + z + \ldots < a.$$

The number of cases is now represented by the definite integral

$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} \ldots dz\, dy\, dx,$$

which involves $n - 1$ integral signs.

In like manner, the aggregate of the mth powers of a part (or numerator of the fraction representing the mean value) is the result of integrating the product of the element in this integral by the mth power of any one of the variables. The integral will be found to have the same value, whichever of the variables in the above order of integrations we employ; but it is simplest to employ the variable for which we first integrate.

228. Integrals of this form are readily evaluated by the help of the theorem of Art. 97. Thus, in the case of five parts, we have to evaluate the quadruple integral

$$I = \int_0^a \int_0^{a-x} \int_0^{a-x-y} \int_0^{a-x-y-z} w^m dw\, dz\, dy\, dx. \quad . \quad . \quad (1)$$

The value of the first or inner integral is

$$\frac{1}{m+1}(a-x-y-z)^{m+1}.$$

Putting β for $a-x-y$, the upper limit, the integral next to be evaluated is

$$\frac{1}{m+1}\int_0^{\beta}(\beta - z)^{m+1}dz.$$

By the theorem referred to, this becomes

$$\frac{1}{m+1}\int_0^{\beta} z^{m+1}dz = \frac{\beta^{m+2}}{(m+1)(m+2)} = \frac{(a-x-y)^{m+2}}{(m+1)(m+2)}.$$

In like manner, putting γ for $a-x$, the result of the next integration is

$$\frac{1}{(m+1)(m+2)}\int_0^{\gamma}(\gamma - y)^{m+2}dy = \frac{(a-x)^{m+3}}{(m+1)(m+2)(m+3)},$$

and finally

$$I = \frac{a^{m+4}}{(m+1)(m+2)(m+3)(m+4)}.$$

229. This integral is the aggregate of the mth powers of a part; also, putting $m = 0$, it gives the total number of cases, namely $\frac{1}{2}\frac{1}{4}a^4$; hence we have, for the mean value of the mth power of one of five parts selected at random,

$$M = \frac{1 \cdot 2 \cdot 3 \cdot 4 a^m}{(m+1)(m+2)(m+3)(m+4)}.$$

This gives $\frac{1}{5}a$ for the mean of a part (agreeing with Art. 224), $\frac{1}{1}\frac{1}{5}a^2$ for the mean square of a part, and so on.

In the general case of n parts, we have in like manner, for the whole number of cases,

$$N = \frac{a^{n-1}}{(n-1)!},$$

and, for the mean mth power of a part,

$$M = \frac{(n-1)!\, m!}{(m+n-1)!} a^m.$$

230. The mean value of a product of powers of two or more parts can be readily found by the aid of the equation

$$\int_0^\alpha x^r(\alpha - x)^s dx = \frac{r!\ \alpha^{s+r+1}}{(s+1)\ldots(s+r+1)} = \frac{r!\ s!}{(s+r+1)!}\alpha^{s+r+1}.* \quad (1)$$

We have now to evaluate the definite integral

$$I = \iiint \ldots x_1^p x_2^q \ldots x_{n-1}^r (a - \Sigma x)^s dx_{n-1} dx_{n-2} \ldots dx_1, \quad (2)$$

in which the $n-1$ variables are positive quantities subject to the condition $\Sigma x < a$, and each of the exponents $p, q, \ldots s$ is zero or a positive integer. Denoting the limit in the first integration by α (compare Art. 228), the part of a last written

* See Ex. VII, 16, p. 128.

is $a - \Sigma x = \alpha - x_{n-1}$; hence the first integral has the form given in equation (1). At the next integration, putting β for the upper limit; $\alpha = \beta - x_{n-2}$, substituting, the integral to be evaluated takes the same form, s being replaced by $s + r + 1$, and r by a new exponent; so that the resulting exponent is of the form $s + r + t + 2$. So also at each step, the new value of s, in the application of equation (1), is the sum of the exponents used and the number of preceding integrations.

Consider now the numerical factors introduced at the successive steps. In the denominator, the new factors run from $s + 1$ to the new value of s; therefore these factors constitute the series of increasing natural numbers. In the numerator, the new factor introduced is the factorial of the new exponent. Hence we have finally, for the value of I in equation (2),

$$I = \frac{p!\, q! \ldots s!}{(n - 1 + p + q + \ldots + s)!} a^{n-1+p+q+\ldots+s}.$$

Dividing by the value of N, Art. 229, we find that, when a is divided at random into n parts, the mean value of the product of the pth power of one part by the qth power of another and so on, is

$$M = \frac{(n-1)!\, p!\, q! \ldots s!}{(n - 1 + p + q + \ldots + s)!} a^{p+q+\ldots+s}.$$

231. The mean value of the least part, the next to the least part, and so on to the greatest part, can be found without integration as follows:* Denote the least part by α, the next by $\alpha + \beta$, the next by $\alpha + \beta + \gamma$, and so on. Then α, β, γ, etc., are all positive, and, summing the parts, we have

$$n\alpha + (n-1)\beta + (n-2)\gamma + \ldots = a.$$

The terms in the first member may now be regarded as random parts into which a is divided. Therefore, the mean

* See Whitworth's "Expectation of Parts" in "Choice and Chance" where also an algebraic proof of the value of M in Art. 230 is given.

value of each term is $\frac{a}{n}$; whence $\frac{a}{n^2}$, $\frac{a}{n(n-1)}$, etc., are the mean values of α, β, . . . The mean value of $\alpha + \beta$ is obviously the sum of those of α and β, and so on. Hence we have, for the several mean values,

$$\frac{a}{n}\cdot\frac{1}{n};$$

$$\frac{a}{n}\left(\frac{1}{n}+\frac{1}{n-1}\right);$$

$$\frac{a}{n}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right).$$

. . . . .

These results will be found to agree with those found by integration, in Art. 226, for the case $n = 3$.

Mean Area of a Triangle with Random Vertices.

232. When the quantity whose mean is required depends upon the position of two variable points, the element or differential of the "number of cases" is the product of the elements upon which the two points respectively fall. For example, let us find the mean area of the triangle APQ, Fig. 44, formed by joining the fixed point A on the circumference and two points P and Q taken at random within the circle. It is, for this purpose, convenient to refer the circle to polar coordinates, taking A for the pole, and the tangent to the circle for the initial line. The equation of the circle is then

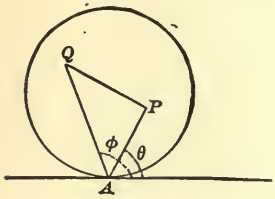

FIG. 44.

$$r = 2a \sin \theta.$$

Denoting by r and θ the polar coordinates of P, and by ρ and

ϕ those of Q, the quantity whose mean is required is the area

$$APQ = \tfrac{1}{2} r\rho \sin(\phi - \theta).$$

The element of the number of cases is now the product of the elements of area upon which P and Q fall, that is,

$$d^4N = r\,dr\,d\theta\,\rho\,d\rho\,d\phi.$$

To avoid negative values of the expression for the area, we suppose $\theta < \phi$, as in Art. 220, which limits the number of cases to one half the product of the areas; thus

$$N = \int_0^{\pi}\int_0^{\phi}\int_0^{2a\sin\phi}\int_0^{2a\sin\theta} r\,dr\,\rho\,d\rho\,d\theta\,d\phi = \tfrac{1}{2}\pi^2 a^4.$$

Using the same limits, the aggregate of the areas APQ is

$$\tfrac{1}{2}\int_0^{\pi}\int_0^{\phi}\int_0^{2a\sin\phi}\int_0^{2a\sin\theta} r^2\,dr\,\rho^2\,d\rho\,\sin(\phi - \theta)\,d\theta\,d\phi.$$

Hence

$$\tfrac{1}{2}\pi^2 a^4 M = \frac{(2a)^6}{18}\int_0^{\pi}\sin^3\phi\int_0^{\phi}\sin^3\theta\,\sin(\phi - \theta)\,d\theta\,d\phi,$$

and

$$M = \frac{64a^2}{9\pi^2}\int_0^{\pi}\sin^3\phi\int_0^{\phi}(\sin\phi\sin^3\theta\cos\theta - \cos\phi\sin^4\theta)\,d\theta\,d\phi.$$

Since

$$\int_0^{\phi}(\sin\phi\sin^3\theta\cos\theta - \cos\phi\sin^4\theta)\,d\theta$$

$$= \tfrac{1}{4}\sin^5\phi - \cos\phi(\tfrac{3}{8}\phi - \tfrac{3}{8}\sin\phi\cos\phi - \tfrac{1}{4}\sin^3\phi\cos\phi),$$

this becomes

$$M = \frac{8a^2}{9\pi^2}\int_0^{\pi}(2\sin^8\phi - 3\phi\sin^3\phi\cos\phi + 3\sin^4\phi\cos^2\phi + 2\sin^6\phi\cos^2\phi)\,d\phi.$$

The value of the integral in this expression is

$$4\frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\frac{\pi}{2} - 3\left[\phi\frac{\sin^4\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^4\phi\,d\phi$$

$$+ 6\frac{3\cdot1}{6\cdot4\cdot2}\frac{\pi}{2} + 4\frac{5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\frac{\pi}{2}$$

$$= \frac{35\pi}{64} + \frac{3}{2}\frac{3\cdot1}{4\cdot2}\frac{\pi}{2} + \frac{3\pi}{16} + \frac{5\pi}{64} = \frac{20+9+6}{32}\pi = \frac{35\pi}{32}.$$

Therefore $M = \frac{8a^2}{9\pi^2}\cdot\frac{35\pi}{32} = \frac{35a^2}{36\pi}$.

233. From this result we can readily derive, by the method of Art. 222, the mean area of the triangle when all three vertices are taken at random within the circle. For, if r is the distance of the point farthest from the centre, the value of N in terms of r is $N = \pi^3 r^6$, whence $dN = 6\pi^3 r^5 dr$; and the mean, M_0, of the new cases (of which dN is the number) is, by the preceding article, $\frac{35r^2}{36\pi}$. Hence, if M denotes the required mean when the radius is a,

$$\pi^3 a^6 M = 6\pi^3\cdot\frac{35}{36\pi}\int_0^a r^7\,dr,$$

whence

$$M = \frac{6}{8}\cdot\frac{35}{36\pi}a^2 = \frac{35a^2}{48\pi}.$$

Mean Areas Found by the Method of Centroids.

234. The theorems proved below are often useful in finding the mean areas of triangles with random vertices. If two vertices, A and B, of a triangle are fixed and the third, C, is taken at random upon a given line or area (C), its area is

$\frac{1}{2}AB \cdot p$, where p is the perpendicular from C upon AB. Supposing that the line or area (C) lies in a plane containing A and B, and furthermore that the line AB or AB produced does not cut (C), the p's will all have the same direction and their mean value will be the perpendicular from G, the centre of gravity of (C). Hence, with this proviso, the mean area of the triangle ABC is that of the triangle ABG.

For example, if A and B are the extremities of a diameter of the circle whose radius is a, and C is taken at random in one of the semicircles, the mean area of ABC is $\frac{4a^2}{3\pi}$, see Art. 206. If C is taken at random on one of the semicircumferences, the mean area is $\frac{2a^2}{\pi}$, see Art. 193. By symmetry, these are also the values when C is taken at random on the whole circle and on the whole circumference respectively.

235. Next suppose that B is not fixed, but taken at random on a line or area (B), lying in the plane which contains (C) and the point A, and also that the line AB can in no position of B cut the area (C); in other words, that no straight line through A can cut both (B) and (C). Then the aggregate of the areas in all the cases in which B has a certain fixed position is $(C) \cdot ABG$, and the aggregate of all values of ABG when B is not fixed is $(B) \cdot AFG$, where F is the centre of gravity of (B). Therefore the aggregate of all values of ABC when neither C nor B is fixed is $(C)(B) \cdot AFG$, and since the total number of cases is $(C) \cdot (B)$, the mean value is AFG.

For example, let the vertex A of a triangle be joined to any point D of the base, and let points P and Q be taken at random respectively in the two parts ABD and ADC. Then, denoting by M_2 the mean of the triangles APQ, we have $M_2 = AFG$, where F and G are the centres of gravity of the two parts. Now, from the known position of the centre of gravity of a triangle, the base FG is $\frac{2}{3}$ of $\frac{1}{2}BC$, and the altitude is $\frac{2}{3}$

of that of the triangle ABC. Hence the mean area required is

$$M_2 = \quad ,$$

where Δ is the area of the triangle ABC.

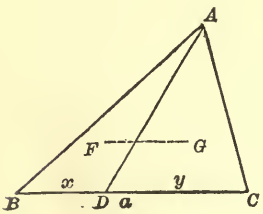

FIG. 45.

236. Certain extensions of this result may now be made. In the first place, let M be the mean area of APQ when P and Q are each taken at random anywhere in ABC. It is plain that for different triangles M is proportional to Δ. Hence putting $BD = x$, $DC = y$, $BC = a$, we have, for the means when P and Q both fall in the triangle whose base is x, or both in the triangle whose base is y, respectively

$$M_x = \frac{x}{a}M, \quad M_y = \frac{y}{a}M.$$

Now putting, for simplicity, $x = y$, we have for the mean when P and Q fall on the same side of AD, $M_1 = \frac{1}{2}M$; and for the mean when they fall on opposite sides, as found above, $M_2 = \frac{2}{9}\Delta$. But since the areas of the parts are now the same, $M = \frac{1}{2}(M_1 + M_2)$; whence $M = \frac{2}{3}M_2$, therefore

$$M = \frac{4}{27}\Delta.$$

237. Next let the triangle have a fixed vertex on a side of the triangle ABC, say at D, Fig. 45. No straight line through D can cut both of the areas ADB and ADC, therefore the condition given in Art. 235 is fulfilled, and denoting as before by M_2 the mean area when one of the other vertices is taken at random in each of the triangles, we now have $M_2 = DFG$, or

$$M_2 = \frac{1}{8}\Delta.$$

Denoting again by M_x and M_y the mean areas when *both* of the other vertices are taken at random in ABD and in ADC

respectively, these means have the same values as in Art. 236, because the fixed vertex is still one of the vertices of the triangle in which the other two fall at random. Hence

$$M_x = \frac{x}{a}\cdot\frac{4}{27}\Delta, \quad M_y = \frac{y}{a}\cdot\frac{4}{27}\Delta.$$

Now denoting by M_D the mean area when the other two vertices are taken at random in ABC, we notice that the whole number of cases is made up of those in which both variable vertices fall in one or other of the two triangles, and those in which one falls in each. Of these classes of cases the means are M_x, M_y and M_2. Since the number of cases in which P falls in ABD and in ADC respectively are proportional to the areas, they are as $x : y$. Hence the numbers of cases mentioned above are as x^2, y^2 and $2xy$; the whole number being thus represented by $(x+y)^2$ or a^2.

It follows that

$$a^2 M_D = x^2 M_x + y^2 M_y + 2xy M_2;$$

hence, substituting the values found above,

$$a^2 M_D = \frac{x^3 + y^3}{a}\frac{4}{27}\Delta + \frac{2xy\Delta}{9}$$

$$= \tfrac{4}{27}\Delta[x^2 - xy + y^2 + \tfrac{3}{2}xy],$$

or, since $y = a - x$,

$$M_D = \frac{2\Delta}{27a^2}(3x^2 - 3ax + 2a^2).$$

Now, to find the mean value when D is taken at random upon the side DC, we have only to integrate this result be-

tween the limits 0 and a. Hence, denoting this new mean by M_0, we have

$$aM_0 = \frac{2\Delta}{27a^2}\int_0^a (3x^2 - 3ax + 2a^2)dx;$$

whence

$$M_0 = \frac{2\Delta}{27}(1 - \tfrac{3}{2} + 2) = \frac{\Delta}{9}.$$

This is obviously also the mean when one vertex is taken at random upon the perimeter and the other two at random within the triangle Δ.

238. The mean M when all three vertices are taken at random within the given triangle may now be found by the method of Art. 222. Let p be the altitude of the given triangle, and z the variable altitude of a similar triangle. Then the area of the variable triangle is kz^2, where k is a constant, and $\Delta = kp^2$. The number of cases when each of the three points is taken at random within the variable triangle is $N = k^3z^6$. Hence the number of new cases introduced when z is changed to $z + dz$ is $dN = 6k^3z^5dz$. The mean M_0 of these new cases has just been shown to be $M_0 = \frac{1}{9}kz^2$. Therefore the equation $MN = \int M_0 dN$ gives

$$k^3p^6M = \tfrac{2}{3}k^4\int_0^p z^7\,dz = \tfrac{1}{12}k^4p^8.$$

Whence

$$M = \tfrac{1}{12}kp^2 = \tfrac{1}{12}\Delta.$$

239. If the three vertices A, B, C of a triangle be taken at random respectively on the given lines or areas (A), (B), (C) in the same plane, but such that no straight line can cut all three, it can be shown as in Art. 235 that the mean area

of the triangle ABC is that of the triangle EFG whose vertices are the centroids of (A), (B) and (C). For example, we can thus find the mean area of a triangle whose vertices are taken at random in any three arcs of the same circumference which do not overlap.

Again, in like manner it may be shown that, if four points be taken at random each within a given volume, the mean volume of the tetrahedron of which they are the vertices is that of the tetrahedron whose vertices are the centroids of the given volumes, provided there is no plane which can cut all four of the given volumes.

Examples XVII.

1. A point falls at random upon a triangle; show that the mean value of its shortest distance from the perimeter is one-third of the radius of the inscribed circle. Show also that the same thing is true of any polygon circumscribed about a circle and of the circle itself.

2. Give the corresponding theorem with respect to a tetrahedron and a sphere.

3. Two points are taken at random on the surface of a sphere; what is their mean distance in a straight line, and also along the surface? $\frac{4}{3}a$; $\frac{1}{2}\pi a$.

4. Find the mean distance of a point taken at random on the surface of a sphere of radius a from the farthest and also from the nearest of the extremities of a fixed diameter.

$\frac{2}{3}(4 - \sqrt{2})a$; $\frac{2}{3}\sqrt{2} \cdot a$.

5. Find the means of the squares of the distances considered in Ex. 4; also show, *à priori*, that their arithmetical mean must be $2a^2$, and is the same thing as the mean of the squared distance of two random points on the surface. $3a^2$; a^2.

6. A number a is divided at random into three parts; find the mean value of their product. $\frac{1}{60}a^3$.

7. Find the mean value of the product of two parts when a is divided at random into four parts. $\frac{1}{20}a^2$.

8. Find the mean distance of all points within a circle of radius a from a fixed point on the circumference. $\frac{32a}{9\pi}$.

9. Determine the average distance between two poirts taken at random within the circle. See Art. 222. $\frac{128a}{45\pi}$.

10. Find the mean distance of a point within a cone of height h and semi-vertical angle α from the vertex. $\frac{\sec^3\alpha - 1}{2\tan^2\alpha}h$.

11. Find the mean distance of a point taken at random within a hemisphere from the opposite pole. $\frac{a}{20}(53 - 16\sqrt{2}) = 1.519a$.

12. Determine the mean value of a chord which cuts a pair of perpendicular diameters of the circle whose radius is a.

$$\frac{32(2-\sqrt{2})}{\pi^2}a = 1.90a.$$

13. Find the mean value of the product of two of the three random parts of a, and the mean value of the cube of this product. $\frac{a^2}{12}$; $\frac{a^6}{560}$.

14. A number a is divided at random into three parts; find the mean square of the least part, of the middle part and of the greatest part. $\frac{a^2}{54}$; $\frac{19a^2}{216}$; $\frac{85a^2}{216}$.

15. Find the mean square of the least of n random parts of a; also the mean pth power. $\frac{2a^2}{(n+1)n^2}$; $\frac{(n-1)!\,p!\,a^p}{(n-1+p)!\,n^p}$.

16. Find the mean value of the square of the next to the least of n random parts of a. $\frac{2(3n^2-3n+1)}{(n+1)n^2(n-1)^2}a^2$.

17. Find the mean area of the right triangle whose hypothenuse is c, all values of either acute angle being equally probable. $\frac{c}{2\pi}$.

18. Find the mean area of the triangle formed by joining the centre and two random points within the circle whose radius is a. $\frac{4a^2}{9\pi}$.

19. Two radii are drawn at random from O, and P and Q are taken at random upon the radii; find the mean area of OPQ. $\frac{a^2}{4\pi}$.

20. P and Q are taken at random one in each of the semicircles into which the diameter $AB = 2a$ separates a circle; find the mean area of APQ. $\frac{4a^2}{3\pi}$.

21. If A is a fixed point of the circumference, and P and Q are taken at random as in Art. 232, show that in $\frac{1}{4}$ of the whole number of cases the centre O is within the triangle APQ. Denoting by M_0 the mean area in these cases, and by M_1 the mean in the remaining cases, thence derive the values of M_0 and M_1 by means of the results of Art. 232 and example 20 above.

$$M_0 = \frac{37a^2}{18\pi}; \quad M_1 = \frac{11a^2}{18\pi}.$$

22. If three points are taken at random in a circle, find the mean area of the triangle of which they are the vertices when the centre is within the triangle, and also when it is not.

$$M_0 = \frac{37a^2}{24\pi}; \quad M_1 = \frac{11a^2}{24\pi}.$$

23. The circumference of a circle is divided into the three parts $2a\alpha$, $2a\beta$, $2a\gamma$, and a point is taken at random in each part; show that the mean area of the triangle of which they are the vertices is

$$\frac{\pi a^2 \sin\alpha \sin\beta \sin\gamma}{2\alpha\beta\gamma}.$$

XVIII.

The Measure of Probability.

240. When it is unknown whether an event has or has not happened (or in the case of a future event, whether it will or will not happen), the probability which we attribute to it depends upon the amount of knowledge we have with respect to its causes. In order to give a numerical value to the probability, there must be a certain set of elementary events or *cases* which, with our present knowledge, we regard as equally probable. In a certain number of these possible cases, the event in question happens; these cases are said to be *favorable* to the event, while the others are *unfavorable*. Then the ratio of the number of favorable cases to the whole number is taken as the measure of the probability of the event.

For example, in a question of throwing dice, the cases which (knowing nothing to the contrary) we assume to be equally probable are the turning up of the different combinations of faces which can be selected from the several dice. With a pair of dice, these are 36 in number. Suppose the event in question to be that of making a throw whose sum is 8; then by actual count there are found to be five favorable cases; hence the probability is $\frac{5}{36}$. Since there are 31 unfavorable cases, this is sometimes expressed by saying that the odds are 31 to 5 *against* throwing 8.

241. When the magnitudes of the quantities concerned in a question of probabilities are continuous, that is capable of indefinite subdivision, the whole number of cases is found by evaluating an integral, exactly as in the treatment of mean values, and the enumeration of the favorable cases is effected in like manner.

For example, suppose that two events are known to have occurred within the same year, and nothing more is known; required the probability that the interval of time between them shall exceed a certain fraction of a year. It is convenient to represent the year by the straight line OA, Fig. 46; then the instant of occurrence of each event is represented by a point taken on OA. Denote the distances from the beginning of the year by x and y, and let $x = OX$ be the greater. If only one point were concerned, the equally probable cases would consist of the falling of the point X upon the several elements dx of the line OA, and the "number of cases" would be represented by the length a of this line.

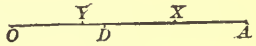

FIG. 46.

But since two points are involved, the elementary case consist of the falling of X and Y upon a particular pair of elements dx and dy. Since we have assumed $y < x$, the number of cases is limited to

$$\int_0^a \int_0^x dy\, dx = \tfrac{1}{2}a^2.$$

The "event" whose probability we seek is that the distance YX shall exceed a given quantity c, equal, say, to OD in Fig. 46. There are no favorable cases when $x < c$; but, for each value of x between c and a, favorable cases occur whenever $y < x - c$. Hence the number of favorable cases is

$$\int_c^a \int_0^{x-c} dy\, dx = \int_c^a (x - c)dx = \frac{(a - c)^2}{2}.$$

Dividing by the whole number, we have for the probability P

$$P = \frac{(a - c)^2}{a^2}.$$

It may be noticed that it is more convenient, in this problem, to find the probability of the distance *exceeding* c, because

the number of cases in which it is less than c can not be expressed by a single definite integral.

Probabilities Represented by Areas.

242. When two variables occur, it is often useful to represent them as in Art. 225 by the rectangular coordinates of a point which thus falls at random within an area which represents the whole number of cases. The number of favorable cases will then be represented by a portion of this area, and may often be found without integration. Thus, in the example above, the whole number of cases is represented by the half OAC of the square in Fig. 47, constructed on the line OA. Taking $OD = c$, the line DE parallel to OC is the locus of the equation $y = x - c$. Hence the favorable area is the area below this line, that is the triangle DAE.

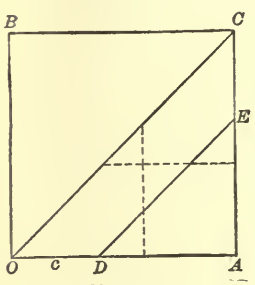

FIG. 47.

In this graphic method the area favorable to the contrary event is exhibited at the same time; thus $ODEC$ is the area favorable to a distance XY less than c, which, as remarked in the preceding article, is not expressible by a single integral.

243. The method is particularly useful when there are conditions which still further restrict the favorable area. For example, we have seen in Art. 223 that the points X and Y in Fig. 46 divide the line a into three parts at random. Suppose now that we require *the probability that no one of the three parts shall exceed c*. The three parts are denoted here by y, $x - y$ and $a - x$. The condition that $x - y$ shall not exceed c cuts off from the whole area OAC, as we have just seen, the triangle ADE. The condition that y shall not exceed c cuts off in like manner the triangle above the horizontal dotted

line, and the condition that $a - x$ shall not exceed c excludes that to the left of the vertical dotted line. The area excluded is in each case $\frac{1}{2}(a - c)^2$; but when (as represented in Fig. 47) $c < \frac{1}{2}a$, these areas overlap, and the remaining area is the small isosceles right triangle whose area is readily seen to be $\frac{1}{2}(3c - a)^2$ (which vanishes when $c = \frac{1}{3}a$, as should evidently be the case). Thus the probability is $\frac{(3c - a)^2}{a^2}$ when c is between $\frac{1}{3}a$ and $\frac{1}{2}a$; and it is $1 - 3\frac{(a - c)^2}{a^2}$ when $c > \frac{1}{2}a$.

Local Probability.

244. Probability questions concerned with the random fall of actual points, lines and other geometrical magnitudes are sometimes called problems of *local probability*. One of those earliest solved was proposed by Buffon in 1777 as follows: A floor of indefinite extent is ruled with equidistant parallel lines whose common distance is a; a rod of length c is thrown upon it at random; what is the chance that it crosses one of the lines?

It is plain that one end of the rod may be assumed to fall upon a fixed line perpendicular to the parallel lines. Denoting by x its distance from one of the intersections taken as origin, and by θ the inclination of the rod to this line, as in Fig. 48, all values of x and of θ are equally probable. But considerations of symmetry show that we need consider only values of x between 0 and a, and values of θ between 0 and $\frac{1}{2}\pi$. Hence we may take $dx\,d\theta$ as the elementary case, and the whole number of cases is

$$\int_0^{\frac{\pi}{2}}\int_0^a dx\,d\theta = \frac{\pi a}{2}.$$

245. In counting the favorable cases, we might begin by assuming a value of x, taken at random, to be fixed; and, after determining the number of favorable cases consistent with this value of x, sum up these results for all possible values of x. Or we might choose θ as the variable to be regarded at first as fixed. In making this choice, we are in fact deciding on the order of integration. In this case, the result is simpler if we regard θ at first as fixed. Now, selecting a value of θ, we see that favorable cases occur when x is between 0 and $c \cos \theta$, the limiting values represented in the lower part of Fig. 48; and if $c < a$, this upper limit is never greater than a, the upper limit used in finding the whole number of cases. Hence, when $c < a$, the number of favorable cases is

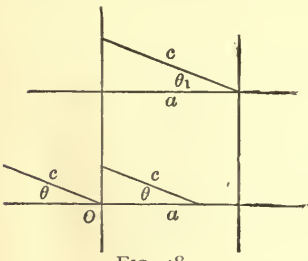

FIG. 48.

$$\int_0^{\frac{\pi}{2}} \int_0^{c \cos \theta} dx \, d\theta = \int_0^{\frac{\pi}{2}} c \cos \theta d\theta = c;$$

and, dividing by the whole number of cases, the probability is

$$P = \frac{2c}{\pi a}.$$

But if $c > a$ there will be a value of θ, as shown in the upper part of the figure, such that, for any smaller value of θ, *all* values of x give favorable cases. The integration for θ must now be separated into two parts at this point, say $\theta_1 = \cos^{-1} \frac{a}{c}$. The number of favorable cases is now

$$\int_0^{\theta_1}\int_0^{a} dx\, d\theta + \int_{\theta_1}^{\frac{\pi}{2}}\int_0^{c\cos\theta} dx\, d\theta = a\theta_1 + c\int_{\theta_1}^{\frac{\pi}{2}} \cos\theta\, d\theta$$

$$= a\theta_1 + c(1 - \sin\theta_1) = a\cos^{-1}\frac{a}{c} + c - \sqrt{(c^2 - a^2)},$$

which gives

$$P = \frac{2}{\pi}\cos^{-1}\frac{a}{c} + \frac{2c}{\pi a} - \frac{2\sqrt{(c^2 - a^2)}}{\pi a}.$$

The limiting value of this expression when c increases without limit is of course unity.

246. When a problem of local probability involves a line drawn from a given point in a random direction in space, it is necessary to assume, as in Art. 199, a spherical surface having the given point for centre, and to regard the line as piercing this surface in a point taken at random upon it. For example, a shot is fired with a given velocity in a random direction from a point on the circumference of a circular field of which the diameter $2a$ is equal to the maximum horizontal range. Given the formula

$$R = 2a\sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the horizontal range, or distance at which the shot falls when θ is the angle of elevation, it is required to determine the chance that the shot may fall within the field.

Let O, Fig. 49, be the point of projection, and let the vertical plane through the line of fire make the angle ϕ with the diameter OA of the field. Then ϕ and θ are the spherical coordinates of the random point on the surface of a sphere of arbitrary radius b. The element of surface is

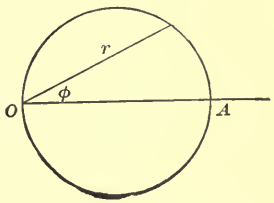

FIG. 49.

$$d^2S = b^2\cos\theta\, d\phi\, d\theta,$$

and the whole surface, representing all possible cases, is that

of the upper hemisphere $2\pi b^2$. Of this area one-half, πb^2, corresponding to values of ϕ on the second and third quadrants, is unfavorable. A portion of the remaining half is also unfavorable and we proceed to find its value.

For a value of ϕ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, as in the diagram, those values of θ are unfavorable which make $R > r$, the radius vector of the field, which is

$$r = 2a \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Comparing with equation (1), we have therefore an unfavorable case if $\sin 2\theta > \cos \phi$. Now, $\sin 2\theta = \cos \phi$ when

$$\theta = \frac{\pi}{4} \pm \frac{\phi}{2};$$

and between these limits (which include 45°, the elevation for maximum range) $\sin 2\theta > \cos \phi$, hence $R > r$. Therefore the additional unfavorable area is

$$S = b^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4} - \frac{\phi}{2}}^{\frac{\pi}{4} + \frac{\phi}{2}} \cos \theta \, d\theta \, d\phi$$

$$= 2b^2 \int_0^{\frac{\pi}{2}} \left[\sin\left(\frac{\pi}{4} + \frac{\phi}{2}\right) - \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \right] d\phi$$

$$= 2\sqrt{2}b^2 \int_0^{\frac{\pi}{2}} \sin \tfrac{1}{2}\phi \, d\phi = 4\sqrt{2}b^2 \cos \tfrac{1}{2}\phi \Big]_{\frac{\pi}{2}}^{0} = 4\left(\sqrt{2} - 1\right)b^2.$$

Therefore the total unfavorable area is $b^2[\pi + 4(\sqrt{2} - 1)]$, and the favorable area is

$$b^2[\pi - 4(\sqrt{2} - 1)].$$

Thus the chance required is $\frac{1}{2} - \frac{2(\sqrt{2} - 1)}{\pi}$, which is about .24.

The Element of Probability.

247. If the element which, when integrated between different limits, gives the whole number of cases and also the number of favorable cases, be first divided by the whole number of cases, the result is *an element of probability*. If this element be integrated with the limits proper to the whole number, it of course gives unity; and if integrated with the limits proper to the favorable number, it gives the probability at once. Thus, when every value of x between 0 and a is equally probable, the element of probability is dx/a. We cannot speak of this as the probability of a special value of x; it may however be called the probability of a value between x and $x + dx$. In the case of the random fall of the point P, it is the chance that P falls upon the element dx of the line a.

248. Now suppose that a point P falls at random within a circle of radius a, and let r denote its distance from the centre. All values of r between 0 and a are now possible, but it is plain that they are not equally probable. The chance that r falls between r and $r + dr$ is now the chance that P falls upon the elementary annulus $2\pi r\,dr$; therefore, dividing by the area of the circle, it is

$$\frac{2r\,dr}{a^2}.$$

Hence the probability in question is proportional, not to dr, but to $r\,dr$;* that is, it is of the form $kr\,dr$. But the value of k depends upon the extreme values between which r is known to fall, and may be determined by the condition that the value

* In using polar coordinates, the probability of r for a point falling at random is proportional to $r\,dr$ *when* θ *has a fixed value*, and that of θ is $d\theta$ when r has a fixed value. Accordingly the probability of the joint occurrence of a particular value of r and a particular value of θ is proportional to the product $r\,dr\,d\theta$.

of the integral between these limits must be unity. Thus, if values of r between r_1 and r_2 only are possible, assuming the form $kr\,dr$ and determining k, we find, for the element of probability,

$$\frac{2r\,dr}{r_2^2 - r_1^2}.$$

In like manner, if a point falls at random within a spherical surface of radius a, its distance from the centre has the elementary probability

$$\frac{3r^2\,dr}{a^3}.$$

249. When the element of probability of one of the variables involved in a problem is independent of the other variables, and can be written down beforehand, the problem can be made to depend upon the simpler case in which the variable in question has a fixed value. Denote this variable by r, and let dp be its element of probability, which we suppose a known function of r and dr. Let P denote the required probability, and P_0 the probability when the value of r is fixed. We suppose P_0 to be first determined; it will of course be a function of r. Then the product $P_0 dp$ will express the probability that the event will happen in connection with a value of r between r and $r + dr$. But this is only *one way* in which the event may happen, for it may happen in connection with any one of the possible values of r. Hence the entire probability of the event is found by summing up the probabilities of the different ways, that is by integration. Thus

$$P = \int P_0\,dp$$

taken between the limits given for r. The method is analogous to that given for mean values in Art. 222.

250. As an illustration let us find the probability that the distance between two points A and B taken at random within a sphere shall be less than the radius a. The probability will evidently be unaltered if we assume one of the points A to be taken on a fixed radius CD of the sphere. Let r denote its distance CA from the centre. The value of r is independent of the other variables (which concern the position of the other point, B) and its elementary probability is, as mentioned in Art. 248,

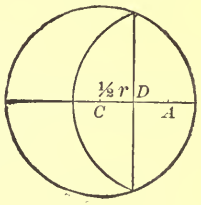

FIG. 50.

$$dp = \frac{3r^2\,dr}{a^3}.$$

Let us now find P_0, the probability that $AB < a$ when A has the position given in Fig. 50.

Let a spherical surface whose centre is A and radius a be described, then P_0 is the chance that B shall fall in the lens-shaped volume cut from the given sphere by this spherical surface. In other words, it is the ratio of the volume of the lens to that of the sphere. The lens is double the segment of the sphere cut off by a plane perpendicular to and bisecting CA at D. Thus its volume is

$$2\pi\int_{\frac{1}{2}r}^{a}(a^2 - x^2)dx = 2\pi(\tfrac{2}{3}a^3 - \tfrac{1}{2}a^2r + \tfrac{1}{24}r^3);$$

and, dividing by the volume of the sphere, $\frac{4}{3}\pi a^3$, we have

$$P_0 = 1 - \frac{12a^2r - r^3}{16a^3}.$$

Therefore

$$dP = P_0\,dp = \frac{3r^2\,dr}{a^3} - \frac{3}{16a^6}(12a^2r^3 - r^5)dr;$$

and, integrating between 0 and a,

$$P = 1 - \tfrac{3}{16}(3 - \tfrac{1}{6}) = 1 - \tfrac{17}{32} = \tfrac{15}{32}.$$

Thus the odds are 17:15 in favor of a distance *greater* than a.

Curves of Probability.

251. The elementary probability of a variable x of which the possible values are not equally probable may be put in the form $y\,dx$, where y is a function of x. The curve in which y is the rectangular ordinate corresponding to the abscissa x (or rather that part of it corresponding to possible values of x) is called *the curve of probability* for the variable x. Such a curve is a graphic representation of *the law of probability* of x, as it may be called, that is the mode in which the probability of x varies. The ratio of any two values of y gives the *relative probability* of the corresponding values of x, although we cannot assign actual values to the probabilities without introducing the element dx. Since the whole probability that x shall fall between given limits is a value of the integral

$$\int y\,dx,$$

the probability that x shall fall between any given limits is represented by the area enclosed between the curve, the axis of x and the ordinates of the limiting values. For this purpose, the total area, of which the base represents the whole range of possible values of x, must of course be taken as unity.

252. In illustration, let us find the law of probability, and construct the probability curve, for the distance from one end of a line $AB = a$ of *the nearer* of two random points which have fallen upon AB. Let P and Q be the two points which fall at random; let X denote the one nearer to A, and x the distance AX. The chance that P shall fall upon any given element dx is dx/a. If this happens, the chance that P is the point X is the same

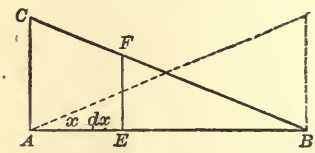

FIG. 51.

thing as the chance that Q falls on the segment $a - x$ to the right of dx. Thus the chance that P falls on dx, *and is the point* X, is

$$\frac{(a - x)dx}{a^2}.$$

But there is an equal chance that Q falls upon dx and is the point X. Therefore the whole chance that X falls upon dx, or value of $y\,dx$, is in this case

$$\frac{2(a - x)dx}{a^2}; \quad \ldots\ldots\ldots \quad (1)$$

hence the probability curve for the nearer to A of two random points is the oblique straight line

$$y = \frac{2(a - x)}{a^2},$$

which passes through B as represented in Fig. 51. Thus the law of probability, in this example, is a uniform decrease of probability from a maximum at A to zero at B. The ordinate AC corresponding to $x = 0$ is $2/a$; this makes the whole area of the triangle ABC unity, as it should be.

253. The probability that x shall fall between any given values is now represented by a part of the area of this triangle. Thus the probability that it shall be less than AE in Fig. 51 is the area of the trapezoid $AEFC$. It is therefore the same thing as the probability that a point falling at random upon the triangle ABC shall fall upon this trapezoid. So also, in general, the probability curve defines an area such that the variable for which it is constructed may be regarded as the abscissa of a point falling at random upon the area.

In particular, if we draw an ordinate which divides the area into two equal parts, the corresponding abscissa is that value which x is just as likely to exceed as to fall short of. This

value is often called *the probable value*. It is, of course, not generally the most probable value. In this case, in fact, any smaller value has a greater relative probability.

254. In like manner, the dotted line in Fig. 51 is the probability curve for the distance of the farther from A of the two random points. Its equation is

$$y = \frac{2x}{a^2}, \quad \text{hence} \quad \frac{2x\,dx}{a^2}$$

is the element of probability. That is, the probability of the distance x of the more distant of two points selected at random is proportional to $x\,dx$. The probable value, in this case, is $a\sqrt{\frac{1}{2}}$; that is to say, it is an even wager that the greater of the distances of the two points shall exceed this value. The alternative event, in this case, is that *both* distances shall fall short of $a\sqrt{\frac{1}{2}}$, the probability of which is also equal to $\frac{1}{2}$.

255. We have seen in Art. 223 that, when two random points fall upon the line a, the distance x of the nearer point from one end is one of three random parts of the line; hence the expression found in Art. 252 and the line CB in Fig. 51 express the law of probability of one of the parts when a is divided at random into three parts. By inspection of the figure, it is evident that the chance that the part shall be less than $\frac{1}{2}a$ is $\frac{3}{4}$; the chance that it shall exceed $\frac{1}{3}a$ (its mean value) is $\frac{4}{9}$; and so on.

256. If three points fall at random upon the line AB, and X, Y, Z denote them when selected in order of nearness to A, the probability curves for their distances from A will be found to be

$$y = \frac{3(a-x)^2}{a^3}; \quad y = \frac{6(a-x)x}{a^3}; \quad y = \frac{3x^2}{a^3}.$$

The proof is similar to that in Art. 252. Thus, for the middle

point: The probability that the points fall in the order P, Q, R, and that Q falls on a given element dx, is the probability of the joint event that Q falls on dx, R falls on the segment $a - x$ to the right of it, and P on the segment x to the left of it. The respective probabilities of these events are

$$\frac{dx}{a}, \quad \frac{a-x}{a} \quad \text{and} \quad \frac{x}{a}.$$

Hence the probability of the compound event is

$$\frac{(a-x)x\,dx}{a^3}.$$

But this is only one way in which the middle point can fall on dx, for there are six orders in which P, Q and R may fall; hence the whole probability that X falls upon dx is

$$\frac{6(a-x)x\,dx}{a^3}.$$

Taking A as before for the origin, the three curves are the parabolas shown in Fig. 52. The first, CB, for the point nearest to A, is also the probability curve for the value of one of *four* parts into which a is divided at random. It will be noticed that the sum of the three elementary probabilities is $\frac{3dx}{a}$; as should be expected, since the sum of the chances that X, Y and Z, respectively, shall fall upon a given dx is evidently the same as the sum of the chances that P, Q and R shall fall upon dx.

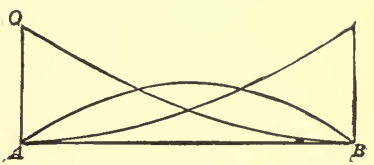

FIG. 52.

Discontinuous Curves of Probability.

257. Since the ordinates of a probability curve are merely graphic representations of the relative probabilities of the corresponding values of x, any curve in which the ordinates are proportional to these relative probabilities will serve as the curve of probability. Then, as mentioned in Art. 253, the equally probable cases correspond to the falling at random of a point upon equal elementary areas of the probability curve. Thus the area whose base is a given range of values represents the number of favorable cases; and, to obtain the numerical value of the probability, this must be divided by the whole area which represents the whole number of cases, or, as we may say, the probability unity.

258. The probabilities of the values of the variable x will frequently, as a result of given conditions, follow different laws in different parts of the range of its possible values; in other words, the probability will be a discontinuous function of x.

When the variable is represented by one coordinate of a point which falls at random upon an area defined by limits of integration, this area will at once determine the law of probability for all values. For example, in Art. 226 we saw that the restrictions upon x and y, which are necessary to make x represent the least, and $a - y$ the greatest, of three random parts into which a is divided, limit the point (x, y) to the triangular area OEF in Fig. 43. Thus the least part is the abscissa of a point falling at random upon this triangle. Inspection of the diagram shows that no value of x greater than $\frac{1}{3}a$ is possible. If a line parallel to the axis of y be drawn corresponding to any smaller value of x, the segment of it included within the triangle measures the relative probability of that value of x. Thus the diagram shows that the probability of a given value of the least part decreases uniformly

from a maximum at the value zero to nothing at the value $\frac{1}{3}a$. No discontinuity occurs, in this case, between the extreme possible values.

259. Now consider the greatest part, which was represented by $a - y$. This is the distance of the point (x, y) from the line BC. The diagram shows that the possible values lie between $\frac{1}{3}a$ and a. The relative probability of a given value is measured by the segment within the triangle of a line parallel to the axis of x at the given distance from BC. The maximum probability occurs, therefore, at the value $\frac{1}{2}a$. From this maximum the probability decreases uniformly to zero at the value a, and it also decreases uniformly to zero as we pass from $\frac{1}{2}a$ to $\frac{1}{3}a$. Thus the curve of probability, when laid down upon the axis of x, is the broken line NPA in Fig. 53. From this figure it is readily shown that the probability that the greatest of three parts exceeds $\frac{1}{2}a$ is $\frac{3}{4}$, the probability that it exceeds $\frac{3}{4}a$ is $\frac{3}{16}$, and so on.

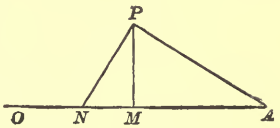

FIG. 53.

Law of Facility of Errors of Observation.

260. In the Theory of Errors of Observation, the probability of the occurrence of an error x is assumed to be proportional to $e^{-h^2x^2}$, where h is a constant depending upon the precision of the observations. The law implies (as indicated by the form of the curve $y = ce^{-h^2x^2}$ shown in Fig. 54) that positive and negative errors numerically equal have the same probability, that the maximum probability occurs at $x = 0$, and that it becomes so small for large values

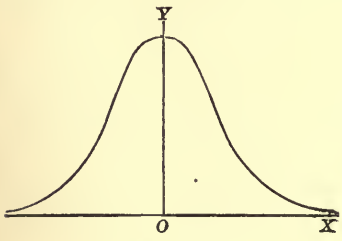

FIG. 54.

of x that it is practically unnecessary to assign any finite limits to the possible errors. Since the number of errors between x and $x + dx$ which may be expected in "the long run" is proportional to $ce^{-h^2x^2}dx$, the curve is often said to express *the law of frequency of errors;* or *the law of facility of the error* x.

In order that $ce^{-h^2x^2}dx$ shall be equal, and not simply proportional, to the probability, it is necessary that the whole area of the curve shall be unity, or

$$c\int_{-\infty}^{\infty} e^{-h^2x^2}dx = 1,$$

which can be shown to give $c = \frac{h}{\sqrt{\pi}}$. See Art. 289.

Mean Values under Given Laws of Probability.

261. In equation (2), Art. 189, p_1, p_2 etc. express the relative frequencies with which the values z_1, z_2 etc. of a variable z occur among the values of which the mean is required. Dividing by Σp, the whole number of cases, the equation becomes

$$M = \frac{p_1}{\Sigma p}z_1 + \frac{p_2}{\Sigma p}z_2 + \ldots \quad ,$$

in which the coefficients of the several values are their probabilities. Denoting these by P_1, P_2 etc., we have

$$M = P_1z_1 + P_2z_2 + \ldots = \Sigma Pz. \quad . \quad . \quad . \quad (1)$$

Thus the mean value under a given law of probability may be defined as the sum of the products of the several values each multiplied by its own probability.

In finding the mean of a continuous variable z, the element which takes the place of p, when integration takes the place of summation, expresses the relative frequency or probability of

the various values of z, in the distribution for which M is required. Denoting this element by dp, the formula for the mean is

$$M \cdot \int dp = \int z\, dp \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which we may regard dp as an element *proportional* to the probability of z. But, when the actual element of probability dP is employed (so that $\int dP = 1$ when the integration corresponds to the whole range of values of z for which M is required), the formula becomes

$$M = \int z\, dP. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

262. For example, in the theory of errors *the mean error*, denoted by ϵ, is defined as that whose square is equal to the mean square of an error. Hence, given the law of frequency, $dp = e^{-h^2x^2}\, dx$, we have

$$\epsilon^2 \cdot \int_0^\infty e^{-h^2x^2} dx = \int_0^\infty x^2 e^{-h^2x^2} dx.$$

Integrating by parts, we have

$$\int_0^\infty x^2 e^{-h^2x^2} dx = -\frac{x}{2h^2} e^{-h^2x^2} \Big]_0^\infty + \frac{1}{2h^2} \int_0^\infty e^{-h^2x^2} dx.$$

in which the first term vanishes at each limit. Hence

$$\epsilon^2 = \frac{1}{2h^2}, \quad \text{or} \quad \epsilon = \frac{1}{h\sqrt{2}}.$$

Again, the mean value of the error without regard to sign is, in the same theory, denoted by η; hence, using the exact element of probability, which (see Art. 260) is $dP = \frac{h}{\sqrt{\pi}} e^{-x^2h^2} dx$, we have

$$\eta = 2\frac{h}{\sqrt{\pi}} \int_0^\infty x e^{-h^2x^2} dx = -\frac{1}{h\sqrt{\pi}} e^{-h^2x^2} \Big]_0^\infty = \frac{1}{h\sqrt{\pi}}.$$

It will be noticed that ϵ is the radius of gyration of the area in Fig. 54 about the axis of y, and η is the abscissa of the centre of gravity of the area on the right of that axis.

Probabilities Involving Selected Points.

263. In any question of probabilities involving a variable x whose values are not equally probable, but follow the same known law throughout its possible values, the element expressing this law takes the place of the simple element dx. Thus, suppose that, in the problem of Art. 241, Y instead of being taken at random is the farthest from O of *three* points taken at random, and that X is the farther from O of *two* other points taken at random; required as before the chance that the distance YX shall exceed c.

Assuming that $y < x$, as in Fig. 46, we proceed as in Art. 241, except that, in accordance with Arts. 254 and 256, dx is replaced by $x\,dx$, and dy by $y^2\,dy$. Thus the whole number of cases is now.

$$\int_0^a\int_0^x y^2\,dy\,x\,dx = \tfrac{1}{3}\int_0^a x^4\,dx = \frac{a^5}{15};$$

and, for the number of favorable cases, we have

$$\int_c^a\int_0^{x-c} y^2\,dy\,x\,dx = \tfrac{1}{3}\int_c^a (x-c)^3 x\,dx.$$

Putting, for convenience, in this last integral $z = x - c$, it becomes

$$\tfrac{1}{3}\int_0^{a-c} z^3(z+c)dz = \frac{1}{3}\,\frac{z^4(4z+5c)}{20}\Big]_0^{a-c} = \frac{(a-c)^4}{60}(4a+c).$$

Hence the probability is $P_1 = \dfrac{(a-c)^4(4a+c)}{4a^5}$.

But if $y > x$, the probability takes a different form. The whole number of cases is now

$$\int_0^a \int_0^y x\, dx\, y^2\, dy = \tfrac{1}{2} \int_0^a y^4\, dy = \frac{a^5}{10},$$

and the favorable number is

$$\int_c^a \int_0^{y-c} x\, dx\, y^2\, dy = \frac{(a-c)^3}{60}(6a^2 + 3ac + c^2);$$

hence the probability is $P_2 = \dfrac{(a-c)^3}{6a^5}(6a^2 + 3ac + c^2)$.

264. If we require the probability that the distance shall exceed c, without distinction of the cases when $y < x$ and $y > x$, we must compare the sum of the favorable cases with the total numbers. The sum of the latter is $\frac{1}{6}a^5$, which is in fact the value of

$$\int_0^a \int_0^a y^2\, dy\, x\, dx.$$

The sum of the favorable cases will be found to reduce to

$$\frac{(a-c)^3 a^2}{6}.$$

Hence the probability that the distance without regard to sign shall exceed c is

$$P_3 = \frac{(a-c)^3}{a^3}.$$

265. If, in the problem solved above, we represent x and y by the rectangular coordinates of a point, as in Art. 242, the point (x, y) is restricted to the square $OACB$, Fig. 55. The total number of cases represented by the integral of the pre-

ceding article may now be regarded as a large number of points distributed not uniformly over the area, but in such a way that the number falling upon any element of surface $dy\,dx$ is $y^2x\,dy\,dx$. In other words, the points are distributed with a *density* proportional to the value of y^2x. The point (x, y) must now be considered as one taken at random from this large number of points; and the probability that it comes from a certain favorable area, determined by the limits of integration, is the ratio of the number of points in the favorable area to the whole number.

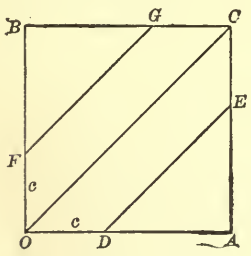

FIG. 55.

Thus, in the present problem, if lines DE and FG parallel to the diagonal OC be drawn cutting off OD and OF each equal to c, the favorable area when $y < x$ is the triangle DAE. Hence the probability P_1, found in Art. 263, is the ratio of the number of points in DAE to that in OAC. In like manner, P_2 is the ratio of the number in FBG to that in OBC, and finally P is the ratio of the number in the two triangles to that in the entire square.

Selected Points upon an Area.

266. We have seen, in Art. 254, that the selection of the more distant from a fixed point of two points taken at random upon a straight line is equivalent to giving it a probability proportional to the distance. Suppose now that two points fall at random upon a circular area of radius a, and that we select the more distant from the centre; let us find the probability that it falls upon a given element of area.

Denoting the random points by P and Q, and by X that which is farther from the center O, the probability that P falls

upon a given element dS of surface is $\frac{dS}{\pi a^2}$, and the probability that P is X is then the probability that Q falls on the area nearer to O than dS, which bears to the whole circle the ratio $\frac{r^2}{a^2}$. Hence, doubling the product, because Q may fall upon dS and be X, the whole probability that X falls on dS is

$$\frac{2r^2 dS}{\pi a^4} = \frac{2r^3 dr\, d\theta}{\pi a^4}.*$$

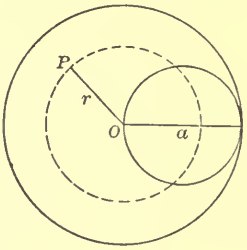

FIG. 56.

267. To find the probability that X falls upon any given area traced out upon the circle, we have only to integrate this over the given area. For example, if the given area is the circle $r = a \cos\theta$ in the diagram, having for diameter one of the radii of the large circle, we have

$$P = \frac{4}{\pi a^4}\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r^3 dr\, d\theta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{3}{16}.$$

The odds are therefore 13 : 3 that the more distant of the two points shall fall outside of the small circle.

Random Lines.

268. A straight line is said to be *drawn at random* in a plane if all directions are equally probable, while, among the lines having a given direction, all points of intersection with a common

* The points X are here distributed with a density proportional to r^2, but the values of r have a probability proportional to r^3. The result would apply to any area upon which points had this distribution, but it is to be noticed that selection of the more distant from a fixed point of two points upon an area not a circle would not produce this distribution except within a circle whose radius is the least value of r upon the boundary of the given area.

perpendicular are equally probable. In expressing the whole number of lines which cut a given convex curve, let p be the perpendicular let fall upon the line from some fixed point *within* the curve, and let ϕ be the inclination of this perpendicular to some fixed direction. Then the whole number will be found by the integration of $dp\,d\phi$. If, in this integration, the limits taken for p are zero and the perpendicular upon a tangent to the curve, ϕ must take all values from 0 to 2π. Thus the number of lines which cut the curve will be

$$\int_0^{2\pi}\int_0^{p} dp\,d\phi = \int_0^{2\pi} p\,d\phi,$$

where p is the perpendicular upon the tangent to the curve.

269. Now it is shown in Diff. Calc., Art. 348 (and is geometrically evident on drawing the figure) that, if τ is the part of the tangent intercepted between the point of contact and the foot of the perpendicular,

$$d\tau = ds - p\,d\phi,$$

where s is the length of the arc measured from some fixed point. Now when we have completed the whole circuit of the curve, so that τ returns to its original value, the integral of $d\tau$ is zero; it follows that, denoting by L the whole length of the curve, we have

$$L = \int_0^{2\pi} p\,d\phi.$$

Therefore L, the length of the curve, is the measure of the number of lines drawn at random which meet the curve. Thus, if a convex curve of length l is drawn within a convex curve of length L, the chance that a random line which cuts L shall also cut l is l/L.

270. If the given line is not convex, or if it is not closed, the same result holds if L denotes the length of the shortest convex closed line which surrounds the given line like an elastic band stretched about it; for it is evident that a straight line which cuts the given line must cut such a band.

For example, the number of lines which pass between two given points whose distance is c is measured by $2c$. Thus, if two points on the circumference of a circle subtend an angle α at the centre, the chance that a random line cutting the circle shall cut both of the arcs into which it is divided by the points is $(2 \sin \frac{1}{2}\alpha)/\pi$.

Probabilities involving Variable Magnitudes.

271. When the probability of an event depends upon a variable magnitude in such a way that the probabilities corresponding to any values of the variable are proportional to the values themselves, *the actual probability of the event is the same as that corresponding to the mean value of the variable.*

For, by hypothesis, the probability that the event will happen when z has the value z_1, z_2, . . . and M are respectively

$$\frac{z_1}{a}, \quad \frac{z_2}{a}, \ldots \quad \text{and} \quad \frac{M}{a};$$

where a is a constant.

Now, dividing equation (1), Art. 261, by a, we have

$$\frac{M}{a} = P_1 \frac{z_1}{a} + P_2 \frac{z_2}{a} + \ldots \ldots \ldots \quad (1)$$

Since P_1 is the probability of the value z_1, the first term of the second member is the chance that z shall have the value z_1, and that the event shall then happen. But this is only one way in which the event can happen, and the second member is the sum of the probabilities of its happening in all possible ways, that is, the total probability of the event. Hence the equation expresses the proposition to be proved.

272. In particular, if a in equation (1) is a fixed area and z is a variable area within it, the chance that a point following at random upon a shall fall upon z is M/a, where M is the mean value of the area z. For example, let Δ be the area of a given triangle ABC, and z that of the triangle PQR, where P, Q and R are three points taken at random on ABC; then we have found in Art. 238 that the mean value of PQR is $\frac{1}{12}\Delta$. It follows that if three points be taken at random on ABC, the chance that a fourth point taken at random on ABC shall fall within PQR is $\frac{1}{12}$. Hence, also, if four points are taken at random, the chance that one of the four shall fall within the triangle found by the other three is $\frac{1}{3}$.* On the other hand, the chance that they shall form the vertices of a convex quadrilateral is $\frac{2}{3}$.

Conversely, the mean value of an area may sometimes be found by knowing the probability that a point falls upon it. Thus, in the foregoing illustration, the sides of the triangle PQR when produced separate the whole triangle ABC into six other parts beside the triangle PQR. If S is a fourth point taken at random, the chance that P falls within the triangle QRS is the same as the chance that S falls within PQR. Hence the mean area of the space in the vertical angle of the angle at P is also $\frac{1}{12}\Delta$. We thus have four areas each of whose mean values is $\frac{1}{12}\Delta$, and it readily follows that each of the other three has the mean value $\frac{2}{3}\Delta$.

273. Since the chance that a given line shall be cut by a random line chosen from a given set of random lines is proportional to its length, the chance of cutting a variable line is the same as the chance of cutting its mean value. For example,

* *If* P, Q, R and S are the points "P falls within QRS," "Q falls within PRS," etc. are mutually *exclusive* events; that is, no two of them can happen at once. The probability that some one of such a set of events shall happen is evidently the sum of their respective probabilities.

a random line is drawn cutting a circle, and two points are taken at random on the circumference; what is the chance that they lie upon opposite sides of the line? This is clearly the same thing as the chance that a random line shall cut the chord joining two random points. The mean value of such a chord was found in Art. 218 to be $M = \frac{4a}{\pi}$; hence (see Art. 270) the chance required is $2M/2\pi a$ or $\frac{4}{\pi^2}$.

274. It should be noticed that the question just solved is not the same thing as the chance that two random secants shall cut one another within the circle. The mean value of the portion of such a secant intercepted by the circle is obviously the double of the mean ordinate to a given diameter which was found in Art. 192 to be $\frac{1}{4}\pi a$. Therefore $M = \frac{1}{2}\pi a$, and the chance in this case is $\frac{2M}{2\pi a} = \frac{1}{2}$.

The mean value of a "chord" in the sense employed above, that is when the line or secant of which it is part (not the extremities of the chord) is taken at random, admits of a simple expression for any convex curve. For, the number of such chords is

$$\iint dp\, d\phi = L;$$

and, if C is the length of the variable chord,

$$L \cdot M = \iint C dp\, d\phi$$

determines the mean value. Now, in this integral, we may take as limits for p the values of the two perpendiculars corresponding to the same value of ϕ (one of which will be negative if the origin is taken within the curve as in Art. 268), provided ϕ varies only between the limits 0 and π. Then the

value of the integral of Cdp will be A, the area of the convex curve, independently of the value of ϕ. Hence we have

$$M = \frac{\pi A}{L}.$$

This gives, for the chance that two such random chords intersect, that is that two random secants intersect within the area,

$$P = \frac{2M}{L} = \frac{2\pi A}{L^2}.$$

Examples XVIII.

1. A floor is ruled with parallel lines at distances $2a$, and also with another set at distances $2b$ perpendicular to them; a rod of length $2c$ less than either of the distances is thrown upon the floor at random. What is the probability that it crosses a line?

$$\frac{c}{\pi ab}(2a + 2b - c).$$

2. Three points are taken at random on the circumference of a circle; what is the chance that no diameter can be drawn having all three points on one side of it? $\frac{1}{4}$.

3. Show that the chance that one of three random parts of a is between c, and $a-c$ is the same as that for one of two random parts.

4. What is the chance that the middle point of three random points falling on a line shall fall on the middle third of the line?

$\frac{1}{2}\frac{3}{7}$.

5. Two points are taken at random in the northern hemisphere; show that the probability that their difference of latitude exceeds α is

$$P = \cos\alpha - (\tfrac{1}{2}\pi - \alpha)\sin\alpha.$$

6. Two points are taken at random upon a semicircle and their ordinates drawn; find the chance that a point taken at random upon the diameter shall fall between the ordinates. $\frac{4}{\pi^2}$.

7. Two points are taken at random within a circle of radius a; find the probability that their distance shall exceed the radius.

$$\frac{3\sqrt{3}}{4\pi}.$$

8. If Y is the farther from O of two points taken at random upon a line OA of length a, and X the farther of two other points taken at random, what is the probability that YX shall exceed c?

$$\frac{(a-c)^2(3a+c)}{a^4}.$$

9. Supposing X in the preceding problem to be the point farther from A of the second pair, find the chance that the distance YX shall exceed c: first, when the points are known to fall in the order $OYXA$; second, when the order is $OXYA$; third, when the order is not known.

$$P_1 = \frac{(a-c)^4}{a^4}; \qquad P_2 = \frac{(a-c)^2(a+c)(5a+c)}{5a^4};$$

$$P = \frac{(a-c)^2(3a^2+2ac+c^2)}{3a^4}.$$

10. What is the chance that the distance exceeds c, if X and Y are the farthest from O respectively of two and of four points?

$$P = \frac{(a-c)^3}{15a^3}(15a^2 - 3ac + 6ac^2 + 2c^3).$$

11. Two points are taken at random in the northern hemisphere; find the elementary probability of the smaller of their latitudes.

$$2(1-\sin\phi)\cos\phi\, d\phi.$$

12. A line a is divided into three parts at random, and that of intermediate value is taken; what is its most probable value, and what the "probable value" in the sense explained in Art. 253?

$$\tfrac{1}{3}a\,; \quad .289a.$$

13. What is the probability that Y, the nearer to the centre of the two points in Art. 267, falls upon the given area? $P = \frac{5}{16}$.

14. A point is taken at random within a circle whose radius is a, and a line is drawn at random through it; find the chance that it cuts a concentric circle whose radius is $c < a$.

$$P = \frac{2\sin^{-1}\frac{c}{a}}{\pi} + \frac{2c\sqrt{(a^2-c^2)}}{\pi a^2}.$$

15. If, in Ex. 14, the line is drawn at random, what is the probability?

$$\frac{c}{a}.$$

16. If two points are taken at random, one on the surface and the other within a sphere of radius r, find the probability that their distance shall be less than c, supposing $c < 2r$. $P = \frac{c^3(8r - 3c)}{16r^4}$.

17. If P is the probability that the distance between two points taken at random within a sphere shall be less than c, and P_0 that found in Ex. 16, show that $d(NP) = P_0 dN$ (compare Art. 222), and thence find the value of P. $P = \frac{c^3(32r^3 - 18cr^2 + c^3)}{32r^6}$.

18. Find the mean distance of the middle of three points taken at random on a line a from the middle of the line. $\frac{3}{16}a$.

19. If a point Z is taken at random upon a line $OA = a$, and then a point X is taken at random on OZ, determine the probability curve for $OX = x$; find also its mean value, and that of its square. $y = \frac{1}{a}\log\frac{a}{x}$; $\frac{a}{4}$; $\frac{a^2}{9}$.

20. Lines of length b and b' fall at random upon a line a; find the chance that they overlap by an amount less than c, where c is less than either b or b', and $a + c > b + b'$. $\frac{c(2a - 2b - 2b' + c)}{(a - b)(a - b')}$.

21. A and B are inhabitants of a city which is known to be situated on a river. Assuming, in default of any knowledge, that the river divides the number of inhabitants into two parts at random, if it is known that B lives on the right bank, what is the probability that A lives on that bank? $\frac{2}{3}$.

22. If, in the preceding example, m inhabitants are known to live on one side, and n on the other, show that the odds that A lives on the first-mentioned side are $m + 1 : n + 1$.

23. A line crosses a circle at random; find the chance that a point taken at random within the circle shall be distant from the line by more than the radius a of the circle. $\frac{2}{3\pi}$.

24. Two random chords of a circle, that is chords whose extremities are taken at random, are drawn; what is the chance that they intersect? $\frac{1}{3}$.

25. Two random lines cut a square; what is the chance that they intersect within the square? $\frac{\pi}{8}$.

26. A line is drawn at random across a circle; what is the chance that two points taken at random within the circle shall lie on opposite sides of it? $\frac{128}{45\pi^2}$.

27. Two points A and B are taken at random in a triangle; what is the chance that two other points taken at random in the triangle shall fall on opposite sides of AB? $\frac{7}{18}$.

28. Four points are taken at random within a circle; what is the chance that they form the vertices of a convex quadrilateral?

$$1 - \frac{35}{12\pi^2}.$$

CHAPTER V.

Definite Integrals.

XIX.

Differentiation of a Definite Integral.

275. The general symbol for a definite integral of a single independent variable is

$$\int_a^b f(x)dx;$$

where a and b are constants, that is to say, independent of x. Denoting its value by u, we have seen in Art. 82 that

$$u=F(b)-F(a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $F(x)$ is such a function that

$$\frac{dF(x)}{dx}=f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

provided $F(x)$ varies continuously while x passes from the value a to the value b. Moreover, this condition will be fulfilled if $f(x)$ is itself one-valued, finite and continuous for the same range of values of x. The independent variable x is used only in defining the integral which, by equation (1), is a function not of x but of the limits. It may be called the *current variable*

when other variables are also under consideration, and it is evidently immaterial what symbol is used for the current variable.

We have, in Section VII, derived the values of certain definite integrals by means of formulæ of reduction which took simple forms by virtue of special values of the limits. We shall in this chapter consider other methods by which such integrals can be evaluated in cases where, for the most part, the value of the indefinite integral, $F(x)$, cannot be obtained.

276. Regarding the upper limit in

$$u = \int_a^b f(x)dx$$

as variable, we have from equations (1) and (2)

$$\frac{du}{db} = F'(b) = f(b), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and in like manner for the lower limit

$$\frac{du}{da} = -F'(a) = -f(a). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the limits were functions of some other variable z, we should have (see Diff. Calc., Art. 371)

$$\frac{du}{dz} = \frac{du}{db}\frac{db}{dz} + \frac{du}{da}\frac{da}{dz} = f(b)\frac{db}{dz} - f(a)\frac{da}{dz}. \quad . \quad . \quad . \quad (3)$$

277. Next, writing the integral in the form

$$U = \int_a^x u\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as in Art. 84, let us suppose that the quantity u under the integral sign is a function of some other variable α beside the current variable x. Then U is also a function of α, we now have

$$\frac{dU}{dx}=u,$$

whence

$$\frac{d}{d\alpha}\frac{dU}{dx}=\frac{du}{d\alpha}.$$

Now, since differentiation with respect to independent variables is commutative (Diff. Calc., Art. 381), this gives

$$\frac{d}{dx}\frac{dU}{d\alpha}=\frac{du}{d\alpha}.$$

Hence, integrating with respect to x, we may write

$$\frac{dU}{d\alpha}=\int_a \frac{du}{d\alpha}dx+C, \quad \ldots\ldots \quad (2)$$

in which the constant of integration C has a definite value because we have fixed the lower limit of the integral. To find this value we notice that when $x=a$ in equation (1), $U=0$ *independently of the value of* α. Therefore, for this value of x, U is a constant with respect to α, and its derivative assumes the value zero. It follows that, putting $x=a$ in equation (2), we find $C=0$. Hence the equation may be written in the form

$$\frac{d}{d\alpha}\int_a^b udx=\int_a^b \frac{du}{d\alpha}dx; \quad \ldots\ldots \quad (3)$$

which expresses that *an integral can be differentiated with respect to a quantity independent of the current variable and the limits by differentiating the expression under the integral sign.*

If the limits were also functions of α, the total derivative with respect to α would also contain the terms given in equation (3) of the preceding article.

278. By means of this theorem, we can derive from the known value of a definite integral the values of a series of other integrals. For example, the first of the fundamental integrals, p. 8, gives, when $n > -1$,

$$\int_0^1 x^n dx = \frac{1}{n+1}; \quad \ldots\ldots\ldots \quad (1)$$

whence, by taking successive derivatives with respect to n, we find

$$\int_0^1 x^n \log x \, dx = -\frac{1}{(n+1)^2}, \quad \ldots\ldots\ldots \quad (2)$$

$$\int_0^1 x^n (\log x)^2 dx = \frac{1.2}{(n+1)^3}; \quad \ldots\ldots\ldots \quad (3)$$

and in general

$$\int_0^1 x^n (\log x)^r dx = (-1)^r \frac{r!}{(n+1)^{r+1}}, \quad \ldots\ldots \quad (4)$$

in which $n+1$ is positive, and r is a positive integer.

Integration under the Integral Sign.

279. Supposing, as in Art. 277, that u is a function of α as well as of x, the definite integral

$$U = \int_a^b u \, dx$$

is a function of α, and $Ud\alpha$ may be integrated between any limits α_0 and α_1 which are admissible in accordance with the conditions given in Art. 82. Thus

$$\int_{\alpha_0}^{\alpha_1} Ud\alpha = \int_{\alpha_0}^{\alpha_1}\int_a^b udx\, d\alpha.$$

We have seen in § X. that the order of integration in this double integral may be changed; no change being required in the limits because each pair is independent of the other variable. Hence

$$\int_{\alpha_0}^{\alpha_1} Ud\alpha = \int_a^b\int_{\alpha_0}^{\alpha_1} ud\alpha\, dx;$$

that is to say, *a definite integral can be integrated with respect to a new variable, between constant limits, by integrating the expression under the integral sign between these limits.*

280. Supposing the value of U to be known, we may often derive, by means of this theorem, the values of other definite integrals. For example, the integral employed in Art. 278,

$$\int_0^1 x^n\, dx = \frac{1}{n+1}, \quad \ldots\ldots\ldots \quad (1)$$

is a function of n; multiplying by dn and integrating, we have

$$\int_0^1\int_r^s x^n dn\, dx = \int_r^s \frac{dn}{n+1}. \quad \ldots\ldots \quad (2)$$

Performing the integrations with respect to n,

$$\int_0^1 \frac{x^s - x^r}{\log x} dx = \log\frac{s+1}{r+1}. \quad \ldots\ldots \quad (3)$$

As a particular case, when $s=1$ and $r=0$, we have

$$\int_0^1 \frac{x-1}{\log x}dx=\log 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Application to the Evaluation of Definite Integrals.

281. It follows from the p eceding articles that, if in the case of a definite integral to be evaluated we can, by differentiation or integration with respect to a new variable, *arrive at a known integral*, the reverse process will give the value of the proposed integral.

For example, from Art. 63, p. 80, we derive the two definite integrals

$$\int_0^\infty e^{-mx}\sin nx\,dx=\frac{n}{m^2+n^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\int_0^\infty e^{-mx}\cos nx\,dx=\frac{m}{m^2+n^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which m must be positive, but n can have either sign. Now suppose the integral to be evaluated is

$$u=\int_0^\infty xe^{-mx}\sin nx\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here the quantity under the integral sign is simplified by integration with respect to n. Thus

$$\int udn=-\int_0^\infty e^{-mx}\cos nx\,dx,$$

or, by equation (2),

$$\int u\,dn = -\frac{m}{m^2+n^2}. \quad \ldots\ldots\ldots \quad (4)$$

Hence, taking derivatives with respect to n,

$$u = \frac{2mn}{(m^2+n^2)^2}.$$

The "constant of integration" implied in the indefinite integral of equation (4) is a quantity independent of n: it need not be determined, however, because it would disappear in the subsequent differentiation.

The proposed integral, u in equation (3), is in fact one of a series of integrals which may be derived from the known integrals (1) and (2) by successive differentiation. See example 6 below.

282. On the other hand, the integral

$$U = \int_0^\infty \frac{e^{-mx}\sin nx}{x}\,dx \quad \ldots\ldots \quad (1)$$

is simplified by differentiation with respect to n. Thus

$$\frac{dU}{dn} = \int_0^\infty e^{-mx}\cos nx\,dx = \frac{m}{m^2+n^2} \quad \ldots\ldots \quad (2)$$

by equation (2) of the preceding article. Hence by integration

$$U = \int \frac{m\,dn}{m^2+n^2} = \tan^{-1}\frac{n}{m} + C,$$

where C is independent of n. Now equation (1) shows that $U=0$ when $n=0$; hence, putting $n=0$ in this equation, we have $C=0$; therefore

$$U = \int_0^\infty \frac{e^{-mx}\sin nx}{x}\,dx = \tan^{-1}\frac{n}{m}, \quad \ldots\ldots \quad (3)$$

in which the $\tan^{-1}$ is the primary value, so that the integral has the sign of n.

283. The special case in which $m=0$ deserves particular notice. Putting $m=0$ in equation (3), we find

$$\int_0^\infty \frac{\sin nx}{x}dx = \frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2}$$

according as n is positive or negative.

The graph of the function

$$y=\int_0 \frac{\sin nx}{x}dx,$$

for the case $n=1$, is represented in Fig. 3, p. 110; the value $\frac{1}{2}\pi$ here found is the ordinate of the asymptote. In the more general graph n is the gradient at the origin. The asymptote retains its position when n is varied, except that when n is negative it lies below the axis of x.

I will be found that the method of Art. 282 fails when applied directly to this definite integral. This results from the fact that, although n occurs in it, the expression is not really a function of n.

Employment of Double Integrals.

284. The process illustrated in Art. 282 is equivalent to putting U in the form of a double integral with both pairs of limits constant, and then changing the order of integration. For, denoting the function under the integral sign by u, so that

$$u=\frac{e^{-mx}\sin nx}{x},$$

we found that

$$\frac{du}{dn} = e^{-mx} \cos nx.$$

This, together with the fact that $u = 0$ when $n = 0$, shows that u may be written in the form

$$u = \int_0^n e^{-mx} \cos nx \, dn,$$

and therefore

$$U = \int_0^\infty \int_0^n e^{-mx} \cos nx \, dn \, dx.$$

The order of integration here indicated of course leads back to the original expression for U; but, reversing the order, we have

$$U = \int_0^n \int_0^\infty e^{-mx} \cos nx \, dx \, dn$$

$$= \int_0^n \frac{m \, dn}{m^2 + n^2} = \tan^{-1} \frac{m}{n}.$$

285. The form of a proposed integral containing no variable except x may suggest the introduction of another variable, in order to make this method of evaluation applicable. Thus if the value of the integral

$$\int_0^1 \frac{x - 1}{\log x} dx,$$

given in equation (4), Art. 280, were unknown, its form might suggest the introduction of the variable exponent n, because differ-

entiation of x^n *as an exponential* will cancel the denominator $\log x$. Thus, putting

$$U=\int_0^1 \frac{x^n-1}{\log x}dx,$$

and noticing that $u=0$ when $n=0$, we have

$$U=\int_0^1\int_0^n x^n\,dn\,dx=\int_0^n\int_0^1 x^n\,dx\,dn=\int_0^n \frac{dn}{n+1}=\log(n+1),$$

of which the proposed integral is the special case corresponding to $n=1$.

286. Again, to evaluate

$$\int_0^\pi \frac{\log(1+\cos x)}{\cos x}dx,$$

we may introduce the variable z thus:

$$U=\int_0^\pi \frac{\log(1+z\cos x)}{\cos x}dx.$$

The function u under the integral sign vanishes when $z=0$; hence, taking its derivative with respect to z, we find that U can be put in the form

$$U=\int_0^\pi\int_0^z \frac{dz}{1+z\cos x}dx=\int_0^z\int_0^\pi \frac{dx}{1+z\cos x}dz,$$

in which we suppose $z<1$. Therefore, by formula (G), p. 124,

$$U=\int_0^z \frac{\pi dz}{\sqrt{(1-z^2)}}=\pi\sin^{-1}z;$$

and putting $z=1$, we have for the particular case

$$\int_0^{\pi} \frac{\log (1+\cos x)}{\cos x}\, dx = \frac{\pi^2}{2}.$$

Transformation by Change of Variable.

287. Examples of the transformation of a definite integral by a change of the current variable have already been given (Arts. 95–98). The theorem of Art. 97 is particularly useful in connection with other transformations. As an illustration, take the integral

$$u = \int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta.$$

By Art. 97, we have also

$$u = \int_0^{\frac{\pi}{2}} \log \cos \theta \, d\theta,$$

and adding

$$2u = \int_0^{\frac{\pi}{2}} \log \frac{\sin 2\theta}{2}\, d\theta = \frac{1}{2}\int_0^{\pi} \log \sin \phi \, d\phi - \log 2 \int_0^{\frac{\pi}{2}} d\theta. \quad . \quad (1)$$

But it is easily shown that

$$\int_0^{\pi} \log \sin \phi \, d\phi = 2\int_0^{\frac{\pi}{2}} \log \sin \phi \, d\phi = 2u;$$

hence, substituting in equation (1), we derive

$$u = \int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta = \int_0^{\frac{\pi}{2}} \log \cos \theta \, d\theta = -\frac{\pi \log 2}{2} = -1.089. \quad (2)$$

288. Transformation of a double integral may sometimes be used to evaluate a definite integral. For example, let

$$k=\int_0^\infty e^{-x^2}dx; \quad \ldots\ldots\ldots \quad (1)$$

then we have also

$$k=\int_0^\infty e^{-y^2}dy.$$

Since each of the expressions under the integral sign is independent of the variable contained in the other and of its limits, the product of the equations gives

$$k^2=\int_0^\infty e^{-x^2}\,dx\cdot\int_0^\infty e^{-y^2}\,dy=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy \quad \ldots \quad (2)$$

Regarding x, y and z as rectangular coordinates, this double integral represents the volume included between the planes of reference and the surface whose equation is

$$z=e^{-(x^2+y^2)}.$$

Transforming to the polar coordinates r and θ, where

$$x=r\cos\theta, \qquad y=r\sin\theta,$$

the same volume is represented by

$$k^2=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}r\,dr\,d\theta,* \quad \ldots\ldots \quad (3)$$

* The analytical transformation of the double element $dx\,dy$ into $r\,dr\,d\theta$ is given in Art. 142. The integral in equation (3) represents the limiting value when the integration extends over a quadrant of a circle, and the radius is then made infinite; moreover this limit is found to be finite. The integral in equation (2) represents the limiting value when the integration extends over a rectangle whose sides are made infinite. It is clear that these limiting values must be equal.

and integrating, we have

$$k^2 = -\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-r^2}\Big]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4};$$

therefore

$$k = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

289. A double integral whose value is k^2 may be formed in a different way, leading to another evaluation of k. Putting $x = az$ in equation (1) of the preceding article, we have

$$k = \int_0^{\infty} e^{-a^2z^2} a\,dz = \int_0^{\infty} e^{-a^2x^2} a\,dx. \quad \ldots \quad (1)$$

Again, taking a as the current variable, we may write

$$k = \int_0^{\infty} e^{-a^2} da.$$

If the element of this last integral be multiplied by the constant k, the value of the integral will be multiplied by k; hence, using the value of k given in equation (1), we find

$$k^2 = \int_0^{\infty}\int_0^{\infty} e^{-a^2} e^{-a^2x^2} a\,dx\,da.$$

Reversing the order of integration, we have

$$k^2 = \int_0^{\infty}\int_0^{\infty} e^{-a^2(1+x^2)} a\,da\,dx = \tfrac{1}{2}\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{4};$$

hence, as before, $k=\frac{1}{2}\sqrt{\pi}$. Also, by equation (1),

$$\int_0^\infty e^{-a^2x^2}\,dx=\frac{k}{a}=\frac{\sqrt{\pi}}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

See Art 260.

290. The evaluation of an integral may sometimes be effected by a combination of the processes illustrated in preceding articles. For example, given

$$u=\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Transforming by putting $x=\frac{a}{z}$, we find

$$u=a\int_0^\infty e^{-\left(\frac{a^2}{z^2}+z^2\right)}\frac{dz}{z^2}=a\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}\frac{dx}{x^2}. \quad . \quad . \quad . \quad (2)$$

Now, differentiating with respect to a, we obtain from equation (1)

$$\frac{du}{da}=-2a\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}\frac{dx}{x^2};$$

hence, comparing with equation (2),

$$\frac{du}{da}=-2u \qquad \text{or} \qquad \frac{du}{u}=-2da.$$

Integrating, we have

$$\log u=-2a+c \qquad \text{or} \qquad u=Ae^{-2a}, \quad . \quad . \quad . \quad (3)$$

where A is a constant independent of a. To determine its value, we notice that, when $a=0$, u becomes the integral whose value

is found in Art. 288; therefore, putting $a=0$ in equation (3), we have $A=\frac{1}{2}\sqrt{\pi}$.

Hence

$$u=\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx=\frac{\sqrt{\pi}e^{-2a}}{2}.$$

Also, by equation (2),

$$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}\frac{dx}{x^2}=\frac{\sqrt{\pi}e^{-2a}}{2a}.$$

Substitution of a Complex Value for a Constant.

291. If a complex value is given to one of the constants in an integral of known value, the integral becomes a complex quantity, and we may assume that the real and imaginary parts are represented by the real and imaginary parts of the known value. For example, if in the integral

$$\int e^{ax}\,dx=\frac{e^{ax}}{a}$$

we put $a=-m+in$ (where m is positive), and apply the limits 0 and ∞, we have

$$\int_0^\infty e^{(-m+in)x}\,dx=\frac{e^{-mx}e^{inx}}{-m+in}\Big]_0^\infty=\frac{e^{-mx}(\cos nx+i\sin nx)}{-m+in}\Big]_0^\infty,$$

or

$$\int e^{-mx}(\cos nx+i\sin nx)=\frac{1}{m-in}=\frac{m+in}{m^2+n^2}.$$

Equating separately the real and imaginary parts, we have the two results,

$$\int_0^\infty e^{-mx}\cos nx=\frac{m}{m^2+n^2},\qquad \int_0^\infty e^{-mx}\sin nx=\frac{n}{m^2+n^2},$$

which we have previously derived by the method of parts, see Art. 281.

Again, in the equation

$$\int_0^\infty e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{2a},$$

Art. 289, put $a^2=ic^2$; whence $a=c(\sqrt{\tfrac{1}{2}}+i\sqrt{\tfrac{1}{2}})$, and we find

$$\int_0^\infty(\cos c^2x^2-i\sin c^2x^2)dx=\frac{\sqrt{\pi}}{c\sqrt{2}}\,\frac{1}{1+i}=\frac{\sqrt{\pi}}{2c\sqrt{2}}(1-i).$$

Hence

$$\int_0^\infty\cos c^2x^2\,dx=\frac{\sqrt{\pi}}{2c\sqrt{2}}\qquad\text{and}\qquad\int_0^\infty\sin c^2x^2\,dx=\frac{\sqrt{\pi}}{2c\sqrt{2}}\;;$$

and, if we put $y=c^2x^2$,

$$\int_0^\infty\frac{\cos y}{\sqrt{y}}d\;=\int_0^\infty\frac{\sin y}{\sqrt{y}}dy=\sqrt{\frac{\pi}{2}}.$$

Examples XIX.

1. Derive a series of integrals by successive differentiation of the definite integral $\int_0^\infty e^{-\alpha x}\,dx$. $\qquad\int_0^\infty x^ne^{-\alpha x}=\frac{n!}{\alpha^{n+1}}.$

2. From the fundamental formula (k') (p. 9) derive

$$\int_0^\infty\frac{dx}{\alpha+\beta x^2}=\frac{\pi}{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}};$$

and thence derive a series of integrals by differentiation with reference to α. $\qquad\int_0^\infty\frac{dx}{(\alpha+\beta x^2)^n}=\frac{\pi}{2^n\beta^{\frac{1}{2}}}\cdot\frac{1\cdot3\ldots(2n-3)}{(n-1)!}\cdot\frac{1}{\alpha^{n-\frac{1}{2}}}.$

3. Derive a series of integrals by differentiating the integral used in Ex. 2 with reference to β.

$$\int_0^\infty \frac{x^{2n-2}dx}{(\alpha+\beta x^2)^n} = \frac{\pi}{2^n\alpha^{\frac{1}{2}}} \frac{1\cdot 3\cdot 5\ldots(2n-3)}{(n-1)!}\cdot\frac{1}{\beta^{n-\frac{1}{2}}}.$$

4. Derive an integral from that employed in Exs. 2 and 3 by differentiating twice with respect to β and once with respect to α.

$$\int_0^\infty \frac{x^4dx}{(\alpha+\beta x^2)^4} = \frac{\pi}{32\alpha^{\frac{3}{2}}\beta^{\frac{5}{2}}}.$$

5. Derive an integral from the result of Ex. II., 67, by differentiation.

$$\int_0^\infty \frac{dx}{(x^2+b^2)\,(x^2+a^2)^2} = \frac{\pi(2a+b)}{4a^3b(a+b)^2}.$$

6. Derive an integral by a second differentiation with respect to m of equation (1), Art. 281; also, by means of Ex. XII., 18, Diff. Calc., p. 111, find the result of r differentiations.

$$\int_0^\infty x^2e^{-mx}\sin nx\,dx = \frac{2n(3m^2-n^2)}{(m^2+n^2)^3}.$$

$$\int_0^\infty x^re^{-mx}\sin nx\,dx = \frac{r!}{(m^2+n^2)^{\frac{r+1}{2}}}\sin\left[(r+1)\cot^{-1}\frac{m}{n}\right].$$

7. Derive an integral by differentiating equation (1), Art. 281, with respect to n.

$$\int_0^\infty xe^{-mx}\cos nx = \frac{m^2-n^2}{(m^2+n^2)^2}.$$

A second differentiation gives again the result of Ex. 6.

8. From the definite integral

$$\int_0^\infty e^{-mx}\cos nx\,dx = \frac{m}{m^2+n^2}$$

derive the result of Ex. 7 by differentiation with respect to m, also an integral by a second differentiation.

$$\int x^2e^{-mx}\cos nx\,dx = \frac{2m(m^2-3n^2)}{(m^2+n^2)^3}.$$

The corresponding general integral, which is

$$\int_0^\infty x^r e^{-mx} \cos nx\, dx = \frac{r!}{(m^2+n^2)^{\frac{r+1}{2}}} \cos\left[(r+1)\cot^{-1}\frac{m}{n}\right]$$

may be derived directly from Ex. 6 *by differentiation with respect to* n.

9. Derive an integral by integrating $\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$.

$$\int_0^\infty \left[\tan^{-1}\frac{p}{x} - \tan^{-1}\frac{q}{x}\right]\frac{dx}{x} = \frac{\pi}{2}\log\frac{p}{q}.$$

10. Derive a definite integral by integrating

$$\int_0^\infty e^{-mx} \sin nx\, dx = \frac{n}{m^2+n^2}$$

with respect to n.

$$\int_0^\infty \frac{e^{-mx}}{x}(\cos ax - \cos bx)dx = \frac{1}{2}\log\frac{m^2+b^2}{m^2+a^2}.$$

11. Derive an integral by integrating

$$\int_0^\infty e^{-mx} \cos nx\, dx = \frac{m}{m^2+n^2}$$

with respect to m.

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos nx\, dx = \frac{1}{2}\log\frac{b^2+n^2}{a^2+n^2}.$$

Each of the integrals, for which Exs. 10 *and* 11 *give the differences, of values corresponding to different constants, is separately infinite. The two results together give the more general difference formula*

$$\int_0^\infty \frac{e^{-mx}\cos ax - e^{-nx}\cos bx}{x} dx = \tfrac{1}{2}\log\frac{n^2+b^2}{m^2+a^2}.$$

12. Derive an integral by integration from the result of Ex. II., 67.

$$\int_0^\infty \frac{1}{x}\left[\tan^{-1}\frac{p}{x} - \tan^{-1}\frac{q}{x}\right]\frac{dx}{x^2+b^2} = \frac{\pi}{2b^2}\log\frac{p(q+b)}{q(p+b)}.$$

13. Evaluate the integral $\int_0^{\frac{\pi}{4}} \log(1+\tan\phi)$, using the theorem of Art. 97. $\frac{\pi \log 2}{8}$.

14. $\int_0^{\infty} \frac{x \log x \, dx}{(x^2+a^2)^2}$. $\frac{\log a}{2a^2}$.

15. $\int_0^{\infty} \frac{\tan^{-1}x \cdot dx}{x^2+x+1}$. $\frac{\pi^2}{6\sqrt{3}}$.

16. $\int_0^{\infty} \tan^{-1}\frac{x}{a}\,\frac{x\,dx}{x^4+a^4}$. $\frac{\pi^2}{16a^2}$.

17. $\int_0^{\infty} \log\frac{x^2+a^2}{x^2+b^2}\,dx$. $\pi(a-b)$.

18. $\int_0^{\infty} \log\left[1+\frac{a^2}{x^2}\right]\log x \, dx$. $\pi a(\log a - 1)$.

19. $\int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}dx$. $\frac{\pi}{b}\log\frac{a+b}{b}$.

20. $\int_0^{\pi} \theta \log\sin\theta \, d\theta$. $-\frac{\pi^2 \log 2}{2}$.

21. Prove that

$$\int_0^1 x^{a-1}\cos(b\log x)dx=\frac{a}{a^2+b^2} \quad \text{and} \quad \int_0^1 x^{a-1}\sin(b\log x)dx=\frac{-b}{a^2+b^2}.$$

22. Evaluate $u=\int_0^{\infty} e^{-a^2x^2}\cos 2rx\,dx$.

Integrate $\frac{du}{dr}$ *by parts.* $u=\frac{\sqrt{\pi}}{2a}e^{-\frac{r^2}{a^2}}$.

23. Putting $u=\int_0^{\infty} \frac{a\cos bx}{a^2+x^2}\,dx$, prove by a double integration by

parts that $b^2u=\frac{d^2u}{da^2}$, which is satisfied by $u=Ae^{ab}+Be^{-ab}$. Thence show that, when a and b are positive,

$$\int_0^\infty \frac{\cos bx}{a^2+x^2}dx=\frac{\pi}{2a}e^{-ab}.$$

24. Derive integrals by differentiation and integration of the result of Ex. 23; and thence deduce the equation of Art. 283.

$$\int_0^\infty \frac{x\sin bx}{a^2+x^2}dx=\frac{\pi}{2}e^{-ab}; \qquad \int_0^\infty \frac{\sin bx}{x(a^2+x^2)}dx=\frac{\pi}{2a^2}[1-e^{-ab}].$$

25. Evaluate the integral $\int_0^\infty \frac{\cos bx\,dx}{(a^2+x^2)^2}$. $\frac{\pi}{4a^3}e^{-ab}(1+ab)$.

XX.

Infinite Values of the Function under the Integral Sign.

292. We have seen in Art. 82 that when $f(x)$ is a real and finite one-valued function for all values of x between and including the values a and b, the integral

$$\int_a^b f(x)\,dx$$

has a real, finite and definite value. In fact, under these circumstances, the graph of the indefinite integral, that is the curve

$$y=\int_a f(x)\,dx,$$

(see Art. 85 *et seq.*) is, for this range of values of x, determined by the fact that it passes through the point $(a, 0)$, and that its gradient is given at every point by the equation

$$\frac{dy}{dx} = \tan\phi = f(x).$$

Now when the value of $f(x)$ is infinite at the limit the integral itself *may* increase without limit; but *not necessarily* so, as illustrated by the graphs in Figs. 5 and 6, pp. 112 and 113, where in each case the kn wn value of the indefinite integral shows that it is finite at the critical points where $f(x)$ is infinite. Thus we can, for example, write

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} dx = \frac{\pi}{2},$$

although $f(x)$ is infinite at the upper limit.

293. When the definite integral is regarded as the limit of a sum (see Art. 99) an integral of this kind is generally characterized by the fact that the extreme elements of the sum vanish when we pass to the limit and the number of elements becomes infinite. Consider, for example, the integral

$$\int_0^{\frac{\pi}{4}} \log\tan\phi \, d\phi,$$

in which $\log\tan\phi$ is infinite at the lower limit. Here the first element of the sum of which the integral is the limit is $\Delta\phi \log\tan\Delta\phi$. Passing to the limit when $\Delta\phi$ vanishes, and writing z for $\Delta\phi$, this element takes the indete minate form $z \log\tan z]_{z=0}$, the value of which is found on evaluation to be zero. Thus the given integral is the sum of an infinite number of vanishing elements and admits of a finite value just as in the ordinary case. Compare the integral evaluated in Art. 287.

Cauchy's General and Principal Values.

294. In general, if $f(x)$ is infinite only at the upper limit b, the integral may be regarded as the limiting value when e is diminished without limit of

$$\int_a^{b-e} f(x)\,dx \quad .$$

(where, supposing $b>a$, e is a small positive quantity), and this may have a finite or an infinite value.

In like manner, when $f(x)$ is finite for a and b and for all intermediate values except the single value c, for which it is infinite, Cauchy regarded the integral as the limit of

$$\int_a^{c-\mu e} f(x)\,dx + \int_{c+\nu e}^{b} f(x)dx,$$

when e decreases without limit, μ and ν being positive numbers. If both parts of this expression have finite limits, the integral has a finite value. If both parts become infinite with the same algebraic sign, the integral is infinite; but, if they become infinite with opposite signs, the result takes the indeterminate form $\infty - \infty$, and *may* have a finite value.

In this last case, the limiting value is generally found to depend upon the ratio $\mu : \nu$. This was called by Cauchy *the general value of the integral;* while the special value assumed when $\mu = \nu$ he called *the principal value of the integral.*

295. For example, if a and b stand for positive quantities, $f(x)$ in the integral $\int_{-a}^{b} \frac{dx}{x}$ is infinite for the single intermediate value $x=0$. Here

$$\int_{-a}^{-\mu e} \frac{dx}{x} + \int_{\nu e}^{b} \frac{dx}{x} = \log \frac{\mu e}{a} + \log \frac{b}{\nu e},$$

which takes the form $-\infty+\infty$ when $e=0$; but, by algebraic reduction, the expression becomes

$$\log\frac{b}{a}+\log\frac{\mu}{\nu},$$

which is accordingly the general value of the integral, and putting $\mu=\nu$, we have for *the principal value* $\log b-\log a$.

Integrals with Infinite Limits.

296. An integral with an infinite upper limit is the limit of an integral of the form

$$y=\int_a^x f(x)\,dx, \quad \ldots\ldots\ldots \quad (1)$$

when x is increased without limit. In order that it may have a finite and definite value, the graph of the indefinite integral, represented by equation (1), must have an asymptote parallel to the axis of x. When this is the case, it is necessary that, if $f(x)$ (which is the gradient or value of $\tan\phi$ in the curve) approaches to a definite * limit, that limit shall be zero.

297. On the other hand, $f(x)$ may approach zero as a limit when $x=\infty$, and yet the integral may increase without limit. For example, in the integral

$$y=\int_1^x \frac{dx}{x},$$

* If (x) does not approach a definite limit, the integral may remain finite as x increases without limit and yet not have a definite limiting value. For example, $\sin x$ has no definite value when $x=\infty$, and $\int_0^x \sin x\,dx$ approaches no definite limit as x increases, but its value must lie between 0 and 2.

$f(x)=0$ when $x=\infty$, but the integral (whose value is $\log x$) becomes infinite. The infinite branch of the graph in this case tends to parallelism to the axis of x, but the branch is parabolic, and y increases without limit.

When the integral is regarded as a sum, the first case is analogous to an infinite series of terms decreasing without limit and having a finite sum, and this last case is analogous to a series (like that given in Art. 180, Diff. Calc., p. 176) in which the terms decrease without limit, but the sum nevertheless has no limit.

298. When $f(x)$ is real and finite for all real values of x, and approaches zero both for $x=\infty$ and $x=-\infty$, the graph of the integral may approach an asymptote at each end. In this case the integral

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

has a finite value, as illustrated by Fig. 7, p. 115.

Again, when the indefinite integral becomes infinite both for positive and negative values, so that both parts of the definite integral become infinite, the integral may have a finite value, if these parts become infinite with opposite signs. In this case, the integral must be regarded as the limit of

$$\int_{-\nu h}^{\mu h} f(x)\,dx,$$

where μ and ν are positive numbers and h increases without limit. This limit, like that considered in Art. 294, is in general dependent upon the ratio $\mu:\nu$, and, by assuming $\mu=\nu$, we can obtain a "principal value."*

* In this case, if the negative branch of the graph were folded over on the axis of y, the principal value would be the limit of the vertical distance between the two branches.

Integrals of Certain Rational Fractions.

299. Let us take, for example, the integral corresponding to a pair of imaginary roots $\alpha \pm i\beta$ in the decomposition of a rational fraction (Art. 18). It will be convenient to put the quadratic fraction in the form

$$\frac{2A(x-\alpha)+2B\beta}{(x-\alpha)^2+\beta^2}, \qquad (1)$$

because this is the sum of the linear partial fractions

$$\frac{A+iB}{x-\alpha+i\beta}+\frac{A-iB}{x-\alpha-i\beta}. \qquad (2)$$

Thus we have to consider the integral

$$\int_{-\infty}^{\infty} \frac{2A(x-\alpha)+2B\beta}{(x-\alpha)^2+\beta^2}\,dx.$$

The indefinite integral is

$$A \log[(x-\alpha)^2+\beta^2]+2B \tan^{-1}\frac{x-\alpha}{\beta}$$

The second term is finite at each limit, and the value when the limits are applied is $2B\pi$ But the first term is infinite at each limit, hence its value is the limit, when h is infinite, of

$$\int_{-\nu h}^{\mu h} \frac{2A(x-\alpha)}{(x-\alpha)^2+\beta^2}\,dx = A \log \frac{(\mu h-\alpha)^2+\beta^2}{(\nu h+\alpha)^2+\beta^2}\Big]_{h=\infty} = 2A \log \frac{\mu}{\nu}.$$

Thus h general value of the integral is

$$\int_{-\infty}^{\infty} \frac{2A(x-\alpha)+2B\beta}{(x-\alpha)^2+\beta^2}\,dx = 2A \log \frac{\mu}{\nu}+2B\pi, \qquad (3)$$

and the principal value is $2B\pi$.

300. Let us suppose that, in the integral

$$\int_{-\infty}^{\infty} \frac{f(x)}{\phi(x)} dx,$$

$\phi(x)$ is a rational integral function of which x^{2n} is the highest term, while $f(x)$ is a polynomial of lower degree; also that $\phi(x)=0$ has no real roots. The fraction can be decomposed into n quadratic fractions corresponding to the n pairs of imaginary roots. Thus

$$\frac{f(x)}{\phi(x)} = \frac{2A_1(x-\alpha_1)+2B_1\beta_1}{(x-\alpha_1)^2+\beta_1^2} + \ldots + \frac{2A_n(x-\alpha_n)+2B_n\beta_n}{(x-\alpha_n)^2+\beta_n^2}. \quad (1)$$

Hence by equation (3) above we have

$$\int_{-\infty}^{\infty} \frac{f(x)}{\phi(x)} dx = 2\Sigma A \cdot \log \frac{\mu}{\nu} + 2\Sigma B \cdot \pi. \quad (2)$$

Thus, in general, the integral is infinite at both limits and is indeterminate in value.

Let us now further suppose that $f(x)$ is *at least two units* lower in degree than $\phi(x)$. Then, equating the coefficients of x^{2n-1} in the result of clearing equation (1) of fractions, we have $0=2\Sigma A$. Therefore, in this case, we have

$$\int_{-\infty}^{\infty} \frac{f(x)}{\phi(x)} dx = 2\Sigma B \cdot \pi, \quad (3)$$

the indefinite integral being now finite at each limit.

301. Let us apply this result to evaluate

$$\int_{-\infty}^{\infty} \frac{x^{2m} dx}{x^{2n}+1},$$

where $m<n$. The roots of the equation $x^{2n}+1=0$ are the values

of $(\cos\pi+i\sin\pi)^{\frac{1}{2n}}$, which, by De Moivre's Theorem, are of the form

$$x=\cos\frac{(2k+1)\pi}{2n}+i\sin\frac{(2k+1)\pi}{2n},$$

where k has the $2n$ values, 0, 1, ... , $2n-1$. These are equivalent to the n conjugate pairs,

$$x=\cos\frac{(2k+1)\pi}{2n}\pm i\sin\frac{(2k+1)\pi}{2n},$$

where k has the n values 0, 1, . . . , $n-1$. Now, if he values of A and B corresponding to a pair of roo s are defined by expression (2), Art. 299, $A-iB$ will be he numera or of the partial fraction corresponding to $\alpha+i\beta$. By equation (3), Art. 21, the value of this numerator is

$$A-iB=\frac{f(\alpha+i\beta)}{\phi'(\alpha+i\beta)}$$

Hence, in the present case,

$$A-iB=\frac{(\alpha+i\beta)^{2m}}{2n(\alpha+i\beta)^{2n-1}}=-\frac{(\alpha+i\beta)^{2m+1}}{2n}$$

because $(\alpha+i\beta)^{2n}=-1$. Hence

$$A_k-iB_k=-\frac{1}{2n}\left[\cos\frac{2k+1}{2n}\pi+i\sin\frac{2k+1}{2n}\pi\right]^{2m+1},$$

or, using DeMoivre's Theorem, and putting for abridgment

$$\theta=\frac{2m+1}{2n}\pi,$$

$$A_k-iB_k=-\frac{1}{2n}[\cos(2k+1)\theta+i\sin(2k+1)\theta].$$

Thus $2B_k = \frac{1}{n} \sin (2k+1)\theta$, and by equation (3), Art. 300,

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{n}[\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta].$$

This expression may be put in a more compact form; for, if

$$S = \sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta,$$

we have

$$\begin{aligned} 2S\sin\theta &= 2\sin^2\theta + 2\sin\theta\sin 3\theta + \ldots + 2\sin\theta\sin(2n-1)\theta \\ &= 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \ldots + \cos(2n-2)\theta - \cos 2n\theta \\ &= 1 - \cos 2n \quad = 2\sin^2 n\theta = 2\sin^2(m+\tfrac{1}{2})\pi = 2. \end{aligned}$$

Therefore $S = \operatorname{cosec}\theta$, and

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{n} \operatorname{cosec} \frac{2m+1}{2n}\pi. \quad \ldots\ldots \quad (1)$$

302. Since the function under the integral sign is unchanged when we change the sign of x, it obviously follows from this result that

$$\int_{0}^{\infty} \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{2n} \operatorname{cosec} \frac{2m+1}{2n}\pi. \quad \ldots\ldots \quad (1)$$

This equation admits of some noteworthy transformations. In the first place, putting $y = x^{2n}$, whence $x = y^{\frac{1}{2n}}$, $dx = \frac{1}{2n} y^{\frac{1}{2n}-1} dy$, we have

$$\frac{1}{2n}\int_{0}^{\infty} \frac{y^{\frac{2m+1}{2n}-1}}{1+y} dy = \frac{\pi}{2n} \operatorname{cosec} \frac{2m+1}{2n}\pi;$$

or, putting $\frac{2m+1}{2n}=r$,

$$\int_0^\infty \frac{y^{r-1}dy}{1+y}=\frac{\pi}{\sin r\pi}, \quad \ldots\ldots (2)$$

which contains a single constant r in place of m and n. Since m and n are positive integers, and $m<n$, the value of r must lie between 0 and 1; but, by the law of continuity, it is otherwise unrestricted, because it can be made to approach a given value (commensurable or incommensurable) as near as we choose, by taking sufficiently large values of m and n.

Since $\sin(r-1)\pi=\sin r\pi$, we may, by putting r in place of $1-r$, put equation (2) in the form

$$\int_0^\infty \frac{dy}{(1+y)y^r}=\frac{\pi}{\sin r\pi}. \quad \ldots\ldots (3)$$

In this equation also we must have $0<r<1$.

Again, putting $y=x^p$ in equation (2), p being positive, we have

$$\int_0^\infty \frac{x^{pr-1}}{1+x^p}dx=\frac{\pi}{p\sin r\pi}$$

or, putting $pr-1=q$,

$$\int_0^\infty \frac{x^q}{1+x^p}dx=\frac{\pi}{p}\operatorname{cosec}\frac{q+1}{p}\pi, \quad \ldots\ldots (4)$$

in which p is positive, and q must lie between -1 and $p-1$. This equation in fact includes equation (1), which is therefore not restricted to integral values of m and n.

As a particular case, we may put $q=0$, *provided* $p>1$; thus, when $p>1$,

$$\int_0^\infty \frac{dx}{1+x^p}=\frac{\pi}{p}\operatorname{cosec}\frac{\pi}{p}. \quad \ldots\ldots (5)$$

Frullani's Integral.

303. Suppose that an integral can be put in the form

$$\int_c^\infty \frac{\phi(x)}{x} dx,$$

in which c is positive, and that while x varies from c to infinity $\phi(x)$ does not become infinite, but retains he same sign (say the positive) and approaches a finite limit *not* zero; then, although the function under the integral sign approaches zero, the value of the integral will be infinite when $x = \infty$. For, let A be the *least* value of $\phi(x)$ for the entire range of values of x, then

$$\int_c^x \frac{\phi(x)}{x} dx > \int_c^x \frac{A dx}{x},$$

because every element of the given integral is not less than the corresponding element of the integral in the second member. But, when x increases without limit, the value of the integral in the second member, which is $A(\log x - \log c)$, becomes infinite; hence, *a fortiori*, the value of the given integral is infinite.

304. But the difference between two integrals of the form considered may be finite. Consider, for example, *Frullani's Integral*,

$$U = \int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx,$$

in which a and b are positive, and $\phi(x)$ does not become infinite for any positive value of x. We notice, in the first place, that, supposing $\phi'(0)$ not to be infinite, zero is admissible as a lower limit, because when $x = 0$ the quantity under the integral sign is found on evaluation to have the finite value $(a-b)\phi'(0)$.

Employing the method of Art 284, we have

$$U=\int_0^\infty\int_b^a \phi'(\alpha x)d\alpha\, dx=\int_b^a\int_0^\infty \phi'(\alpha x)dx\, d\alpha.$$

Now

$$\int_0^\infty \phi'(\alpha x)dx=\frac{\phi(\alpha x)}{\alpha}\Big]_0^\infty=\frac{\phi(\infty)-\phi(0)}{\alpha},$$

or

$$U=[\phi(\infty)-\phi(0)]\int_b^a\frac{d\alpha}{\alpha};$$

therefore

$$\int_0^\infty\frac{\phi(ax)-\phi(bx)}{x}dx=[\phi(\infty)-\phi(0)]\log\frac{a}{b}. \quad . \quad . \quad (1)$$

For example, if $\phi(x)=\tan^{-1}x$, $\phi(\infty)=\frac{1}{2}\pi$ and $\phi(0)=0$, therefore

$$\int_0^\infty\frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x}dx=\frac{\pi}{2}\log\frac{a}{b}.$$

Compare Ex. XIX., 9.

305. When $\phi(x)$ is a function which, although remaining finite, has no definite limiting value when $x=\infty$, equation (1) fails to determine the value of he in egral. Fo example, when $\phi(x)=\cos x$, cos ∞ has no definite value. The following mode of investigating the in egral will, however, show that in these cases equat on (1) will hold true if for $\phi(\infty)$ we substitute the mean value of $\phi(x)$ over an infinite ange of values of x.

For this purpose, we put

$$u=\int_0^h\frac{\phi(z)-\phi(0)}{z}dz,$$

an integral having a finite value when h is finite. Putting $z=ax$, we find

$$u=\int_0^{\frac{h}{a}}\frac{\phi(ax)-\phi(0)}{x}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In like manner,

$$u=\int_0^{\frac{h}{b}}\frac{\phi(bx)-\phi(0)}{x}dx.$$

Supposing $a>b$, this last equation can be written in the form

$$u=\int_0^{\frac{h}{a}}\frac{\phi(bx)-\phi(0)}{x}dx+\int_{\frac{h}{a}}^{\frac{h}{b}}\frac{\phi(bx)}{x}dx-\phi(0)\int_{\frac{h}{a}}^{\frac{h}{b}}\frac{dx}{x}. \quad . \quad . \quad (2)$$

Equating the values of u in equations (1) and (2), we have

$$\int_0^{\frac{h}{a}}\frac{\phi(ax)-\phi(bx)}{x}dx=\int_{\frac{h}{a}}^{\frac{h}{b}}\frac{\phi(bx)}{x}dx-\phi(0)\log\frac{a}{b}. \quad . \quad (3)$$

Frullani's Integral is the limit when $h=\infty$ of the integral in the first member. As to that in the second member, let us first suppose that the mean value of $\phi(x)$ over the infinite range of values of x is zero. Since the greatest value of the factor $\frac{1}{x}$ under the integral sign is $\frac{a}{h}$ (which is its value at the lower limit),

$$\int_{\frac{h}{a}}^{\frac{h}{b}}\frac{\phi(bx)}{x}dx<\frac{a}{h}\int_{\frac{h}{a}}^{\frac{h}{b}}\phi(x)dx. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the mean value of $\phi(x)$ over the range of values $\frac{h}{a}$ to $\frac{h}{b}$ is

$$M=\frac{1}{\frac{h}{b}-\frac{h}{a}}\int_{\frac{h}{a}}^{\frac{h}{b}}\phi(x)dx;$$

hence, if the limiting value of M is zero, that of the integral in

the second member of the inequality (4) is zero; and, *a fortiori*, that of the first member is zero. Substituting in equation (3), and then making h infinite, the result is the same as if $\phi(\infty)$ in equation (1), Art. 304, were equal to zero. For example, putting $\phi(x)=\cos x$, the limiting value of M is zero, and we have

$$\int_0^\infty \frac{\cos ax-\cos bx}{x}dx=\log\frac{b}{a}.$$

Finally, if the mean value of $\phi(x)$ for an infinite range of values of x is the finite quantity M, we may put

$$\phi(x)=M+\psi(x),$$

in which $\psi(x)$ is a function having zero for its mean value. Substituting in equation (3), and then making h infinite, we derive

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx=[M-\phi(0)]\log\frac{a}{b}, \quad . \quad . \quad . \quad (4)$$

where M takes the place of $\phi(\infty)$ in equation (1), Art. 304.

306. It may be remarked that when $\phi(\infty)=0$, and also when its mean value is zero, the two parts of Frullani's Integral are not each infinite at the upper limit (as they were under the conditions named in Art. 303). In like manner, if $\phi(0)=0$, they are not infinite at the lower limit. Thus, if both these conditions hold, both zero and infinity are admissible limits of the single integral $\int\frac{\phi(ax)dx}{x}$. But, in this case, the second member of equation (4) vanishes, and we have

$$\int_0^\infty \frac{\phi(ax)}{x}dx=\int_0^\infty \frac{\phi(bx)}{x}dx.$$

It follows that an integral of this form has a value independent of a, so long as a is positive, which has been assumed in our demonstration. This might also have been inferred from the result of substituting z for ax. But even in this last process we cannot infer that the value is the same for positive and negative values of a, because the integral ceases to have a meaning when a by gradual change of value passes through the value zero. An integral of this form may therefore be a *discontinuous function* of a. For example, we have seen in Art. 283 that

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2}$$

according as a is positive or negative.

Integrals obtained by Expansion.

307. We have seen in Art. 78 that, when the integration of $f(x)dx$ cannot be effected in finite terms, the integration, term by term, of its development in powers of x will give the like development of the integral. Thus, if

$$\phi(x) = \int_0 \log \frac{1}{1-x} \cdot \frac{dx}{x}, \quad \ldots\ldots \quad (1)$$

we have, by development of the logarithm,

$$\phi(x) = \int_0 \left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots\right) dx;$$

whence, integrating term by term,

$$\phi(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots. \quad \ldots\ldots \quad (2)$$

This defines $\phi(x)$ by a series which is convergent when $x<1$. It is also convergent when $x=1$, its value (see Diff. Calc., Art. 238) then being $\frac{1}{6}\pi^2$. Thus

$$\phi(1)=\int_0^1 \log\frac{1}{1-x}\frac{dx}{x}=\frac{\pi^2}{6}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

308. In like manner, the development of the quantit under the integral sign in any other form of infinite series may admit of integration term by term. For example, if we transform the integral considered in the preceding article by putting $x=1-z$, we have

$$\phi(1-z)=\int_1 \frac{\log z}{1-z}\,dz,$$

and, expanding the factor $(1-z)^{-1}$ in a series convergent when z is between 0 and 1,

$$\phi(1-z)=\int_1 \log z(1+z+z^2+\ldots)dz. \quad . \quad . \quad . \quad (4)$$

Each term under the integral sign is of the form $z^n \log z\,dz$, and integrating by parts we have

$$\int_1 z^n \log z\,dz=\frac{z^{n+1}\log z}{n+1}-\frac{z^{n+1}}{(n+1)^2}+\frac{1}{(n+1)^2}.$$

Giving to n the values 0, 1, 2, etc., and substituting in equation (4),

$$\phi(1-z)=\log z\left[z+\frac{z^2}{2}+\frac{z^3}{3}+\ldots\right]-\left[z+\frac{z^2}{2^2}+\frac{z^3}{3^2}+\ldots\right]$$
$$+\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right].$$

The coefficient of $\log z$ is the series for $-\log(1-z)$, hence the equation may be written

$$\phi(1-z) = -\log z \log(1-z) - \phi(z) + \phi(1).$$

Putting x for z, we have the relation

$$\phi(x) + \phi(1-x) = \frac{\pi^2}{6} - \log x \log(1-x) \quad . \quad . \quad . \quad . \quad (5)$$

as a property of this function. As a particular result, putting $x = \frac{1}{2}$, we find

$$\phi(\tfrac{1}{2}) = \int_0^{\frac{1}{2}} \log \frac{1}{1-x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}.$$

309. In the case of a definite integral, the result of expansion in a series of integrable form is in general a numerical series, by means of which, if convergent, the value may be computed; while in some cases the sum may be already known. For example, to find

$$U = \int_0^{\pi} \frac{x \sin x \, dx}{+\cos^{2p} x},$$

where p is a positive integer. Expanding $(1+\cos^{2p}x)^{-1}$, we have

$$U = \int_0^{\pi} x \sin x (1 - \cos^{2p}x + \cos^{4p}x - \cos^{6p}x + \dots) dx.$$

Integrating the typical term by parts,

$$\int_0^{\pi} x \cos^{2rp} x \sin x \, dx = -\frac{x \cos^{2rp+1} x}{2rp+1}\Big]_0^{\pi} + \frac{1}{2p+1} \int_0^{\pi} \cos^{2rp+1} x \, dx$$

$$= \frac{\pi}{2rp+1},$$

since the final integral vanishes. Hence, giving to r the values 0, 1, 2, etc., we hav

$$\int_0^\pi \frac{x \sin x\, dx}{1+\cos^{2p} x} = \pi\left(1 - \frac{1}{2p+1} + \frac{1}{4p+1} - \frac{1}{6p+1} + \dots\right).$$

When $p=1$, the series is the result of putting $x=1$ in Gregory's series for $\tan^{-1} x$; thus, as a particular case,

$$\int_0^\pi \frac{x \sin x\, dx}{1+\cos^2 x} = \pi\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{\pi^2}{4}.$$

Series in Sines and Cosines of Multiple Angles.

310. When A is numerically greater than B, the expression $A + B\cos x$ can be reduced to the form $m(1 + 2a\cos x + a^2)$, where a is real and may be taken less than unity. (See Art. 313.) Now the expression $1 + 2a\cos x + a^2$ admits of conjugate imaginary factors, thus

$$1 + 2a\cos x + a^2 = (1 + a\cos x + ai\sin x)(1 + a\cos x - ai\sin x),$$

or, using the exponential notation (Diff. Calc., Art. 222),

$$1 + 2a\cos x + a^2 = (1 + ae^{ix})(1 + ae^{-ix}).$$

The sums and differences of the expansions of like functions of these imaginary factors may, by means of the equations

$$2\cos x = e^{ix} + e^{-ix} \quad \text{and} \quad 2i\sin x = e^{ix} - e^{-ix},$$

be expressed in terms of the sines and cosines of the multiples of the angle x, thus giving rise to a variety of developments in multiple angles. Among the simplest are the series deduced below.

311. From the expansions

$$\frac{1}{1+ae^{ix}} = 1 - ae^{ix} + a^2e^{2ix} - a^3e^{3ix} + \ldots, \quad \ldots \quad (1)$$

$$\frac{1}{1+ae^{-ix}} = 1 - ae^{-ix} + a^2e^{-2ix} - a^3e^{-3ix} + \ldots, \quad \ldots \quad (2)$$

we have by addition

$$\frac{2+2a\cos x}{1+2a\cos x+a^2} = 2[1 - a\cos x + a^2\cos 2x - a^3\cos 3x + \ldots.];$$

whence, subtracting unity to simplify the numerator,

$$\frac{1-a^2}{1+2a\cos x+a^2} = 1 - 2a\cos x + 2a^2\cos 2x - 2a^3\cos 3x + \ldots \quad (3)$$

Again, the difference of equations (1) and (2) gives, after dividing by $2ai$,

$$\frac{\sin x}{1+2a\cos x+a^2} = \sin x - a\sin 2x + a^2\sin 3x - a^3\sin 4x + \ldots \quad (4)$$

These series are convergent when $a<1$.

In like manner, from the logarithmic series,

$$\log(1+ae^{ix}) = ae^{ix} - \frac{a^2}{2}e^{2ix} + \frac{a^3}{3}e^{3ix} - \ldots,$$

$$\log(1+ae^{-ix}) = ae^{-ix} - \frac{a^2}{2}e^{-2ix} + \frac{a^3}{3}e^{-3ix} + \ldots,$$

we have, by addition,

$$\log(1+2a\cos x+a^2) = 2\left[a\cos x - \frac{a^2}{2}\cos 2x + \frac{a^3}{3}\cos 3x - \ldots\right]. \quad (5)$$

Again, by subtraction, we derive from the same equations

$$\log\frac{1+ae^{ix}}{1+ae^{-ix}}=2i\left[a\sin x-\frac{a^2}{2}\sin 2x+\frac{a^3}{3}\sin 3x-\dots\right]. \quad (6)$$

To reduce the first member to the pure imaginary form, we have

$$\frac{1+ae^{ix}}{1+ae^{-ix}}=\frac{1+a\cos x+ai\sin x}{1+a\cos x-ai\sin x},$$

in which let us put $1+a\cos x=\rho\cos y$, $a\sin x=\rho\sin y$, so that

$$\tan y=\frac{a\sin x}{1+a\cos x}.$$

We shall then have

$$\log\frac{1+ae^{ix}}{1+ae^{-ix}}=\log\frac{\cos y+i\sin y}{\cos y-i\sin y}=\log\frac{e^{iy}}{e^{-iy}}=2iy.$$

Hence equation (6) becomes

$$y=\tan^{-1}\frac{a\sin x}{1+a\cos x}=a\sin x-\frac{a^2}{2}\sin 2x+\frac{a^3}{3}\sin 3x-\dots \quad (7)$$

A discussion of this equation will be found in Art. 324 *et seq.*

Integrals Developed in Multiple-angle Series.

312. The direct integration of equation (3) of the preceding article between 0 and x gives, when $a<1$,

$$(1-a^2)\int_0^x\frac{dx}{1+2a\cos x+a^2}$$
$$=x-2\left[a\sin x-\frac{a^2}{2}\sin 2x+\frac{a^3}{3}\sin 3x-\dots\right] \quad . \quad . \quad (1)$$

in which the series is convergent.

An expression, like that in the second member, which is periodic except for one term of the first degree in the independent variable, is sometimes called a *quasi-periodi* function. The integral of a periodic functi n goes through the same states of increase or decrease in corresponding parts of successive periods of the independent variable. Hence, except when the increment corresponding to a complete period vanishes, such an integral must have this quasi-periodic character.*

In the presen case, the series vanishes for o and 2π, and also for the half-period π, so that we have

$$\int_0^{\pi} \frac{dx}{1+2a\cos x+a^2} = \frac{\pi}{1-a^2}. \quad \ldots\ldots \quad (2)$$

When $x=\frac{1}{2}\pi$, the terms containing even powers of a in equation (1) vanish, and we have

$$(1-a^2)\int_0^{\frac{\pi}{2}} \frac{dx}{1+2a\cos x+a^2} = \frac{\pi}{2} - 2\left[a - \frac{a}{3} + \frac{a^5}{5} + \frac{a^7}{7} + \ldots\right],$$

whence, by Gregory's series,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+2a\cos x+a^2} = \frac{1}{1-a^2}\left[\frac{\pi}{2} - 2\tan^{-1} a\right]. \quad \ldots \quad (3)$$

313. When $a>1$, in the integral of equation (1) or in others involving the expression $1+2a\cos x+a^2$, a development in converging series is readily obtained. For, denoting such a value by a', we have

$$1+2a'\cos x+a'^2 = a'^2(1+2a\cos x+a^2),$$

* See Fig. 61, p. 353, for the graph of such a function.

in which a stands for the reciprocal of a' and is therefore less than unity. In this way, equation (1) becomes

$$\int_0^x \frac{dx}{1+2a'\cos x+a'^2}=\frac{1}{a'^2-1}\left[x-2\left\{\frac{\sin x}{a'}-\frac{\sin 2x}{2a'^2}+\frac{\sin 3x}{3a'^3}-\cdots\right\}\right],$$

and equations (2) and (3) become

$$\int_0^{\pi} \frac{dx}{1+2a'\cos x+a'^2}=\frac{\pi}{a'^2-1}$$

and

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+2a'\cos x+a'^2}=\frac{1}{a'^2-1}\left[\frac{\pi}{2}-2\cot^{-1}a'\right],$$

the values of both integrals being essentially positive.

When the more general expression $A+B\cos x$ (in which $A>B$ numerically) occurs, we make, as proposed in Art. 310,

$$A+B\cos x=m(1+2a\cos x+a^2)$$

by putting

$$A=m(1+a^2), \qquad B=2ma.$$

Eliminating m and solving the quadratic for a, we have reciprocal roots which we may write in the form

$$a=\frac{A-\sqrt{(A^2-B^2)}}{B}\,*, \qquad \frac{1}{a}=\frac{A+\sqrt{(A^2-B^2)}}{B};$$

whence also

$$m=\frac{A+\sqrt{(A^2-B^2)}}{2}.$$

* In these equations, A is regarded as positive (which does not affect the generality of the results), and the radical is by hypothesis real. Thus a denotes that root which is numerically less than unity, and its sign is that of B.

For example, we have, from equation (1), Art. 312,

$$\int_0^x \frac{dx}{A+B\cos x} = \frac{1}{m(1-a^2)}\left[x - 2\left\{a\sin x - \tfrac{1}{2}a^2\sin 2x + \tfrac{1}{3}a^3\sin 3x - \ldots\right\}\right]$$

$$= \frac{1}{\sqrt{(A^2-B^2)}}\left[x - 2\left\{a\sin x - \tfrac{1}{2}a^2\sin 2x + \ldots\right\}\right]$$

on substituting the values of m and a above.

This integral has already been expressed in finite form, thus, see formula G, p. 41:

$$\int_0^x \frac{dx}{A+B\cos x} = \frac{2}{\sqrt{(A^2-B^2)}}\tan^{-1}\left[\sqrt{\frac{A-B}{A+B}}\tan\frac{x}{2}\right].$$

The inverse tangent in this expression is therefore a quasi-periodic function, and substituting the values of A and B in terms of a and b we find

$$\tan^{-1}\left[\frac{1-a}{1+a}\tan\frac{x}{2}\right] = \frac{x}{2} - \left[a\sin x - \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x - \ldots\right].$$

314. From equation (5), Art. 311, we have by integration

$$\int_0^x \log(1+2a\cos x+a^2)dx = 2\left[a\sin x - \frac{a^2}{2^2}\sin 2x + \frac{a^3}{3^2}\sin 3x - \ldots\right] \quad (1)$$

when $a<1$. Again, since, in the notation of the preceding article,

$$\log(A+B\cos x) = \log m + \log(1+2a\cos x+a^2),$$

we have

$$\int_0^x \log(A+B\cos x)dx$$

$$= x\log\frac{A+\sqrt{(A^2-B^2)}}{2} + 2\left[a\sin x - \frac{a^2}{2^2}\sin 2x + \ldots\right].$$

Thus the integral takes in general a quasi-periodic form and is a pure periodic function only when it can be put in the form (1) with $a<1$.* The case in which $a>1$ in the integral of equation (1) belongs to the general case, the corresponding value of m being a^2; thus, when $x=\pi$, we have

$$\int_0^\pi \log(1+2a\cos x+a^2)dx = 2\pi \log a \quad \text{or} \quad 0,$$

according as a is greater or less than unity.

315. The series in Art. 311 lead to the immediate evaluation of definite integrals of a certain form. For example, if

$$U=\int_0^\pi \cos rx \log(1+2a\cos x+a^2)dx,$$

where $a<1$ and r is a positive integer, we have by equation (5)

$$U=2\int_0^\pi (a\cos x\cos rx-\tfrac{1}{2}a^2\cos 2x\cos rx+\tfrac{1}{3}a^3\cos 3x\cos rx-\ldots)dx.$$

By Art. 33, every term of the integral vanishes except the rth term. Thus we have

$$U=-2(-1)^r\frac{a^r}{r}\int_0^\pi \cos^2 rx\,dx=-\frac{(-a)^r\pi}{r}.$$

Compare Art. 321.

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* The necessary condition is $A=1+\frac{1}{4}B^2$, with the further restriction that B shall not exceed 2.

Examples XX.

1. Show that $\int_0^x \log \cot x\, dx$ vanishes when $x=\frac{1}{2}\pi$ and has a maximum value when $x=\frac{1}{4}\pi$. Sketch the forms of the graphs of this integral and those of $\int_0^x \log \sin x\, dx$ and $\int_0^x \log \cos x\, dx$. See Arts. 293 and 287.

Evaluate the integrals:

2. $\int_0^\infty \frac{dx}{1+x^4}$, $\frac{\pi\sqrt{2}}{4}$.

3. $\int_0^\infty \frac{dx}{1+x^6}$, $\frac{\pi}{3}$.

4. $\int_0^\infty \frac{x^7\, dx}{1+x^9}$, $\frac{\pi}{9 \sin 20^\circ}$.

5. $\int_0^\infty \frac{dx}{1+x^{\frac{3}{2}}}$, $\frac{4\pi}{3}$.

6. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$, π.

7. $\int_0^\infty \frac{dx}{(1+x^3)\sqrt{x}}$, $\frac{2\pi}{3}$.

8. Show that, when $n>1$,

$$\int_0^\infty \frac{dx}{x^n+x^{-n}} = \frac{\pi}{2n} \sec \frac{\pi}{2n}.$$

9. Evaluate $\int_0^\infty \log(1+2a\cos x+a^2)\frac{dx}{c^2+x^2}$ when $a<1$, employing the result of Ex. XIX., 23. $\frac{\pi}{c}\log(1+ae^{-c})$.

10. Evaluate $\int_0^\infty \frac{1}{c^2+x^2}\frac{dx}{1+2a\cos bx+a^2}$. $(a<1)$.

$$\frac{\pi}{2c(1-a^2)}\frac{1-ae^{-bc}}{1+ae^{-bc}}.$$

11. Show that, if b and c are positive,

$$\int_0^\pi \log(b^2 \pm 2bc\cos x + c^2)\,dx = 2\pi \log k,$$

where k is the greater of the quantities b and c. Hence show by differentiation that

$$\int_0^\pi \frac{b + c\cos x}{b^2 + 2bc\cos x + c^2}\,dx = \frac{\pi}{b} \quad \text{or} \quad 0$$

according as $b > c$ or $b < c$.

12. Evaluate $\int_0^\pi \frac{x \sin x\,dx}{1 + 2a\cos x + a^2}$ when $a < 1$ and when $a > 1$.

$$\frac{\pi}{a}\log\frac{1}{1-a};\quad \frac{\pi}{a}\log\frac{a}{a-1}.$$

13. Evaluate $\int_0^\pi \frac{\cos rx\,dx}{1 - 2a\cos x + a^2}$ where r is a positive integer and $a < 1$.

$$\frac{\pi a^r}{1-a^2}.$$

14. Show that, when $a < 1$,

$$(1-a^2)\int_0^\pi \frac{x\,dx}{1 + 2a\cos x + a^2} = \frac{\pi^2}{2} + 4\left(a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \frac{a^7}{7^2} + \dots\right).$$

15. Derive the expansion

$$\int_0^\pi \tan^{-1}\frac{a\sin x}{1 + a\cos x}\,dx = 2\left(a + \frac{a^3}{3^2} + \frac{a^5}{5^2} + \dots\right);$$

and thence show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Compare Diff. Calc., Art. 238.

16. Evaluate $\int_0^\pi \frac{\sin x \sin rx\,dx}{1 - 2a\cos x + a^2}$ when $a < 1$ and when $a > 1$, r being a positive integer.

$$\frac{\pi a^{r-1}}{2};\quad \frac{\pi}{2a^{r+1}}.$$

17. Show that, when $a<1$,

$$\int_0^\infty \frac{x}{c^2+x^2}\,\frac{\sin x\,dx}{1+2a\cos x+a^2}=\frac{\pi e^{-c}}{2(1+ae^{-c})}.$$

18. Derive the expansion

$$\int_0^\pi x\tan^{-1}\frac{a\sin x}{1+a\cos x}dx=\pi\left(a+\frac{a^2}{2^2}+\frac{a^3}{3^2}+\dots\right)$$

when $a<1$, and verify the result when $a=1$.

19. Derive the expansion, when $a<1$,

$$\int_0^\pi x\log(1-2a\cos x+a^2)dx=4\left(a+\frac{a^3}{3^3}+\frac{a^5}{5^3}+\frac{a^7}{7^3}+\dots\right);$$

and thence deduce

$$\int_0^{\frac{\pi}{2}}\theta\log\sin\theta\,d\theta=-\frac{\pi^2\log 2}{8}+\frac{1}{2}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots\right).$$

20. Evaluate the integral $\int_0^1 \frac{x^p-x^{1-p}}{x(1-x)}dx$, where $0<p<1$.

$\pi\cot p\pi$.

Expand and see Diff. Calc., Ex. XXIII., 20.

XXI.

Functions expressed in Multiple-angle Series.

316. The infinite series involving cosines of the multiples of an angle, namely,

$$C+A_1\cos\theta+A_2\cos 2\theta+A_3\cos 3\theta+\dots,$$

is convergent for all values of θ, provided the absolute values of the coefficients form a convergent series. The series then

expresses *an even periodic function* of θ. Under similar circumstances

$$B_1 \sin\theta + B_2 \sin 2\theta + B_3 \sin 3\theta + \ldots$$

expresses *an uneven periodic function* of θ. In either case, the period is 2π, and the values of the function corresponding to values of θ between 0 and π determine the values of the function for all values of θ.

So also, if we put $\theta = \pi x/l$, the series express even and uneven periodic functions of x, in which $x = l$ corresponds to $\theta = \pi$, so that the values of the function for values of x within the half-period from 0 to l determine all the values of the function.

It is frequently desirable, especially in the physical applications of mathematics, to express a given function $f(x)$ in one of these forms, and it was found by Fourier that, notwithstanding the periodic character of the series, it is possible so to determine the coefficients in either series that the sum of the series shall *for a range of values of x from 0 to l* represent any given function $f(x)$.

Fourier's Series.

317. Let us now assume that it is possible to put

$$f(x) = C + A_1 \cos\frac{\pi x}{l} + A_2 \cos 2\frac{\pi x}{l} + \ldots + A_n \cos n\frac{\pi x}{l} + \ldots, \quad (1)$$

for all values of x between 0 and l;* the values of the coefficients can then be determined as follows:

* The demonstrations of Poisson and Lagrange establish the possibility of equation (1) in a direct manner. In that of Lagrange, the series is at first assumed to consist of a limited number of terms. The coefficients (n in number) are then so determined that the equation is satisfied for n equidistant values of x, subdividing the interval between 0 and l. Thus the graph of the series is made to coincide with that of $f(x)$ at n points. Afterward n is made infinite, so that the graphs coincide for an unlimited number of points.

We have seen in Art. 33, that if m and n are positive integers,

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0 \quad \text{or} \quad \tfrac{1}{2}\pi,$$

according as m and n are unequal or equal. It follows that, putting $\pi x/l = \theta$, if equation (1) be multiplied by $\cos m\theta \, d\theta$, and then the integral of each member be taken between the limits 0 and π for θ, that is between 0 and l for x, every term in the second member will disappear from the result except that in which $n = m$. Thus we have, when $n > 0$,

$$\frac{\pi}{l}\int_0^l f(x) \cos n\frac{\pi x}{l}\, dx = \frac{\pi}{2} A_n;$$

whence

$$A_n = \frac{2}{l}\int_0^l f(x) \cos n\frac{\pi x}{l} dx = \frac{2}{\pi}\int_0^\pi f\left(\frac{l\theta}{\pi}\right) \cos n\theta \, d\theta. \quad . \quad (2)$$

Again, multiplying equation (1) by $d\theta$ and integrating,

$$\frac{\pi}{l}\int_0^l f(x)dx = C\int_0^\pi d\theta = \pi C;$$

whence

$$C = \frac{1}{l}\int_0^l f(x)dx.* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Using a separate symbol for the current variable in the integrals, the result may be expressed thus:

$$f(x) = \frac{1}{l}\int_0^l f(v)dv + \frac{2}{l}\sum_{n=1}^{n=\infty} \cos n\frac{\pi x}{l}\int_0^l f(v) \cos n\frac{\pi v}{l} dv, \qquad (4)$$

which is true for all values of x between 0 and l.

* The absolute term C is in form the same as $\frac{1}{2}A_0$ where A_0 is defined by equation (2), but the case $n = 0$ sometimes fails to be included in the general evaluation of the definite integral.

318. For example, let us put $f(x)=x$. Substituting in equation (2), we have

$$A_n=\frac{2l}{\pi^2}\int_0^{\pi}\theta\cos n\theta\,d\theta=\frac{2l}{n^2\pi^2}\int_0^{n\pi}\phi\cos\phi\,d\phi.$$

Integrating by parts, we find

$$\int_0^{n\pi}\phi\cos\phi\,d\phi=\phi\sin\phi\Big]_0^{n\pi}-\int_0^{n\pi}\sin\phi\,d\phi=\cos n\pi-1=0 \quad \text{or} \quad -2,$$

according as n is even or odd. Therefore $A_n=0$ when n is even, and $A_n=-\frac{4l}{n^2\pi^2}$ when n is odd. Again, equation (3) gives $C=\frac{1}{2}l$. Hence, when $0<x<l$,

$$x=\frac{1}{2}l-\frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos 3\frac{\pi x}{l}+\frac{1}{5^2}\cos 5\frac{\pi x}{l}+\dots\right). \qquad (1)$$

This result holds also for the extreme values 0 and l; as is readily verified by means of the equation

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots=\frac{\pi^2}{8},$$

proved in Diff. Calc., Art. 238.

The complete graph of the second member of equation (1) coincides with the line $y=x$ for the interval between $x=0$ and $x=l$. Since the cosine-series is an even function, the portion between $x=0$ and $x=-l$ is symmetrical to this with respect to the axis of y, and the rest of the graph is a periodic repetition of these parts forming a broken line as represented in Fig. 57.

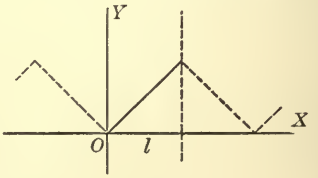

FIG. 57.

The Series in Multiple Sines.

319. In like manner, we may assume

$$f(x)=B_1 \sin\frac{\pi x}{l}+B_2 \sin 2\frac{\pi x}{l}+B_3 \sin 3\frac{\pi x}{l}+\ldots \qquad (1)$$

Then, since, by Art. 33,

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0 \quad \text{or} \quad \tfrac{1}{2}\pi,$$

according as m and n are unequal or equal, multiplying equation (1) by $\sin m\theta \, d\theta$ and integrating between 0 and π, we find

$$B_n=\frac{2}{l}\int_0^l f(x) \sin n\frac{\pi x}{l}dx=\frac{2}{\pi}\int_0^\pi f\left(\frac{l\theta}{\pi}\right)\sin n\theta \, d\theta; \qquad (2)$$

and the result of substitution in equation (1) may be written

$$f(x)=\frac{2}{l}\sum_{n=1}^{n=\infty} \sin n\frac{\pi x}{l}\int_0^l f(v) \sin n\frac{\pi v}{l}dv. \quad . \quad . \quad . \quad (3)$$

The graph of the sine-series thus determined coincides with that of $f(x)$ from $x=0$ to $x=l$, while from $x=0$ to $x=-l$ it forms an arc symmetrical to this with respect to the origin as represented in Fig. 58. Thus, unless $f(0)=0$, the sine-series is a discontinuous function approaching different limits when x approaches zero from one side or the other. The rest of the graph is a periodic repetition of these arcs, and thus, unless $f(l)=0$, the series is discontinuous also at $x=l$. In each case, the value of the series at a point of discontinuity is zero.

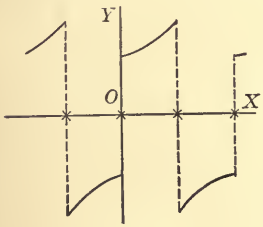

FIG. 58.

320. As an illustration, let us find the sine-series for $f(x)=x$. From equation (2), we have

$$B_n=\frac{2l}{\pi^2}\int_0^{\pi}\theta\sin n\theta\,d\theta=\frac{2l}{n^2\pi^2}\int_0^{n\pi}\phi\sin\phi\,d\phi.$$

Integrating by parts,

$$\int_0^{n\pi}\phi\sin\phi\,d\phi=-\phi\cos\phi\Big]_0^{n\pi}+\int_0^{n\pi}\cos\phi\,d\phi=-n\pi\cos n\pi;$$

whence

$$B_n=-(-1)^n\frac{2l}{n\pi}.$$

Therefore, substituting in equation (1),

$$x=\frac{2l}{\pi}\left(\sin\frac{\pi x}{l}-\frac{1}{2}\sin 2\frac{\pi x}{l}+\frac{1}{3}\sin 3\frac{\pi x}{l}-\ldots\right). \quad (1)$$

The graph of this series is not discontinuous at the origin because the value of $f(0)$ is zero. In fact, in this case, and whenever $f(x)$ is an uneven function, the graph of the series coincides with that of $f(x)$ for the whole interval between $x=-l$ and $x=l$, but in general *exclusively* of these limits. In like manner, when $f(x)$ is an even function, the corresponding cosine-series will represent the function for all values between $x=-l$ and $x=l$ and, in that case, *inclusive* of the limiting values.

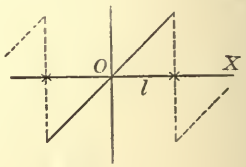

FIG. 59.

321. As a converse application of the equations of Arts. 317 and 319, we may remark that, when the expansion of a function

in sines or cosines of multiples of x has been otherwise obtained, the known values of the coefficients give the values of the definite integrals which occur in those equations. For example, we have seen in Art. 315 that the cosine-series for $\log(1+2a\cos x+a^2)$ gives the value of

$$\int_0^\pi \log(1+2a\cos x+a^2)\cos rx\,dx,$$

where r in an integer, the process employed being the same as that employed in Art. 317 to derive the general expression for the coefficient A_n.

Again, putting $a=1$ in equation (5), Art. 311, we derive

$$\log\cos\tfrac{1}{2}x=-\log 2+\cos x-\tfrac{1}{2}\cos 2x+\tfrac{1}{3}\cos 3x-\ldots,$$

which is convergent for all values of x up to $x=\pi$. From the various coefficients we can therefore infer, when $n=0$,

$$\int_0^\pi \log\cos\tfrac{1}{2}x\,dx=\pi C=-\pi\log 2$$

(which agrees with Art. 287); and also, when n is an integer,

$$\int_0^\pi \log\cos\tfrac{1}{2}x\cdot\cos nx\,dx=\frac{\pi}{2}A_n=-(-1)^n\frac{\pi}{2n}.$$

Developments containing both Sines and Cosines.

322. If, when $f(x)$ is neither an even nor an uneven function, a development in multiple angles applicable from $x=-l$ to $x=l$ be required, it will be necessary to separate $f(x)$ into its even and uneven parts, for development, the one in cosines and the other

in sines. Denoting these by $\phi(x)$ and $\psi(x)$ respectively, as in Diff. Calc., Art. 216, they are

$$\phi(x)=\tfrac{1}{2}\{f(x)+f(-x)\} \quad \text{and} \quad \psi(x)=\tfrac{1}{2}\{f(x)-f(-x)\},$$

so that $f(x)=\phi(x)+\psi(x)$.

Then, because the integral of an even function between the limits $-l$ and l is double the integral between 0 and l, the coefficients for $\phi(x)$ and $\psi(x)$ respectively, Arts. 317 and 319, may be written in the form

$$C=\frac{1}{2l}\int_{-l}^{l}\phi(v)dv, \quad A_n=\frac{1}{l}\int_{-l}^{l}\phi(v)\cos n\frac{\pi v}{l}dv.$$

$$B_n=\frac{1}{l}\int_{-l}^{l}\psi(v)\sin n\frac{\pi v}{l}dv.$$

But, because the integrals between $-l$ and l of the uneven functions $\psi(v)\cos n\frac{\pi v}{l}$ and $\phi(v)\sin n\frac{\pi v}{l}$ vanish, the coefficients may also be written

$$C=\frac{1}{2l}\int_{-l}^{l}f(v)dv, \quad A_n=\frac{1}{l}\int_{-l}^{l}f(v)\cos n\frac{\pi v}{l}dv,$$

$$B_n=\frac{1}{l}\int_{-l}^{l}f(v)\sin n\frac{\pi v}{l}dv.$$

Hence

$$f(x)=\phi(x)+\psi(x)=\frac{1}{2l}\int_{-l}^{l}f(v)dv$$

$$+\frac{1}{l}\sum_{n=1}^{n=\infty}\left(\cos n\frac{\pi x}{l}\int_{-l}^{l}f(v)\cos n\frac{\pi v}{l}+\sin n\frac{\pi x}{l}\int_{-l}^{l}f(v)\sin n\frac{\pi v}{l}\right)dv.$$

323. For example, to obtain a development of e^{mx} true for all values of x between $-l$ and l, we have

$$C=\frac{1}{2l}\int_{-l}^{l} e^{mv}dv=\frac{1}{2ml}(e^{ml}-e^{-ml}),$$

$$A_n=\frac{1}{l}\int_{-l}^{l} e^{mv}\cos n\frac{\pi v}{l}dv=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\frac{ml\theta}{\pi}}\cos n\theta\, d\theta,$$

$$B_n=\frac{1}{l}\int_{-l}^{l} e^{mv}\sin n\frac{\pi v}{l}dv=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\frac{ml\theta}{\pi}}\sin n\theta\, d\theta.$$

To evaluate these integrals, we have, by Art. 63,

$$\int e^{m\theta}\cos n\theta\, d\theta=\frac{e^{m\theta}}{m^2+n^2}(m\cos n\theta+n\sin n\theta),$$

$$\int e^{m\theta}\sin n\theta\, d\theta=\frac{e^{m\theta}}{m^2+n^2}(m\sin n\theta-n\cos n\theta);$$

whence

$$\int_{-\pi}^{\pi} e^{m\theta}\cos n\theta\, d\theta=(-1)^n\frac{m}{m^2+n^2}(e^{m\pi}-e^{-m\pi}),$$

$$\int_{-\pi}^{\pi} e^{m\theta}\sin n\theta\, d\theta=-(-1)^n\frac{n}{m^2+n^2}(e^{m\pi}-e^{-m\pi}).$$

Hence, putting ml/π for m, and substituting,

$$A_n=(-1)^n\frac{ml(e^{ml}-e^{-ml})}{m^2l^2+n^2\pi^2},\quad B_n=-(-1)^n\frac{n\pi(e^{ml}-e^{-ml})}{m^2l^2+n^2\pi^2}.$$

Therefore

$$e^{mx} = ml(e^{ml} - e^{-ml})\left(\frac{1}{2m^2l^2} - \frac{\cos\frac{\pi x}{l}}{m^2l^2+\pi^2} + \frac{\cos 2\frac{\pi x}{l}}{m^2l^2+2^2\pi^2} - \dots\right)$$

$$+\pi(e^{ml} - e^{-ml})\left(\frac{\sin\frac{\pi x}{l}}{m^2l^2+\pi^2} - \frac{2\sin 2\frac{\pi x}{l}}{m^2l^2+2^2\pi^2} + \frac{3\sin 3\frac{\pi x}{l}}{m^2l^2+3^2\pi^2} - \dots\right).$$

The first line is of course the development of cosh mx, and the second that of sinh mx (Diff. Calc., Art. 217). Thus, taking $l=\pi$,

$$\cosh mx = \frac{2m\sinh m\pi}{\pi}\left(\frac{1}{2m^2} + \sum_1^\infty \frac{(-1)^n\cos nx}{m^2+n^2}\right),$$

$$\sinh mx = \frac{2\sinh m\pi}{\pi}\sum_1^\infty \frac{-(-1)^n n\sin nx}{m^2+n^2}.$$

Discontinuity of the Fourier Sine-series.

324. We have seen in Art. 319 that the Fourier sine-series is in general a discontinuous function, presenting sudden changes of value as illustrated by the graphs Figs. 58 and 59. The mode in which an ordinary sine-series, which represents a continuous periodic function, takes on this character is well illustrated by equation (7), Art. 311, namely,

$$y = \tan^{-1}\frac{a\sin x}{1+a\cos x} = a\sin x - \frac{a^2}{2}\sin 2x + \frac{a^3}{3}\sin 3x - \dots, \quad (1)$$

in which $a<1$, so that the series is convergent for all values of x. When $x=0$, the value of the series is zero, so that the symbol $\tan^{-1}$ must then be taken to mean *the primary value* of the inverse tangent. Now, since the denominator, $1+a\cos x$, cannot vanish, y cannot reach either of the values $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$, and therefore the $\tan^{-1}$ is for all values of x restricted to denoting *the primary value* which lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$.

When $a>1$, the series is divergent and equation (1) ceases to have any meaning.

In the intermediate case, when $a=1$, the series remains convergent for all values of x, and becomes, in fact, the Fourier sine-series for $\frac{1}{2}x$, when we take $l=\pi$ in equation (1), Art. 320. Accordingly, the function y now takes the form $\tan^{-1}\tan\frac{1}{2}x$, of which one value is $\frac{1}{2}x$. But the series always represents the primary value of the $\tan^{-1}$ (which is $\frac{1}{2}x$ only when x is between $-\pi$ and π); thus it becomes a discontinuous function, increasing with x, but dropping the value π suddenly whenever x passes through an odd multiple of π.

325. The development of the same function y in a series convergent when $a>1$ is readily obtained from Art. 311, in which

$$2iy=\log\frac{1+ae^{ix}}{1+ae^{-ix}}.$$

For this may be written

$$2iy=\log\frac{e^{ix}}{e^{-ix}}+\log\frac{1+\frac{1}{a}e^{-ix}}{1+\frac{1}{a}e^{ix}},$$

in which, if $a>1$, the second term can be developed as in equation (6). Hence, dividing by $2i$, we have, when $a>1$,

$$y=\tan^{-1}\frac{a\sin x}{1+a\cos x}=x-\left(\frac{\sin x}{a}-\frac{\sin 2x}{2a^2}+\frac{\sin 3x}{3a^3}-\dots\right). \quad (2)$$

The series in this equation is convergent, and is an ordinary periodic function, so that y is, in this case, a quasi-periodic function.

If we now make $a=1$ in equation (2), the value of y becomes x diminished by the Fourier series for $\frac{1}{2}x$, and is again equal to $\frac{1}{2}x$ when x is between $-\pi$ and π; but as x increases, this expression for y receives a sudden increase of π in value when x passes through an odd multiple of π.

Geometrical Illustration.

326. The function y, developed in equations (1) and (2) above, admits of a simple geometrical construction. For, if, in Fig. 60, we take $BA=1$ and $AP=a$, and denote the arcual measure of the angle PAC by x, that of the angle PBC will represent y, where

$$y=\tan^{-1}\frac{a\sin x}{1+a\cos x}.$$

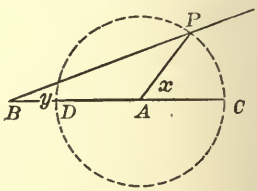

FIG. 60.

As x increases indefinitely, P describes a circle; and, supposing $a<1$, as in the diagram, y is a periodic function returning to a given initial value when x is increased by 2π. The figure represents the "slit-bar" mechanism, in which, when AP is nearly equal to AB, the bar BP vibrates with a "quick-return motion," whereby a quantity nearly equal to π is subtracted from the angle y whenever P passes through a certain small arc in the neighborhood of D.

Now let $a=1$, so that B falls on the circumference of the circle; then we have, for values of x less than π, $y=\frac{1}{2}x$; and we

may take this as universally true, making the bar BP rotate uniformly at half the rate of AP. But the series in equation (1) continues to represent the primary value of the position angle y, which, as x increases, is subject to a periodic dropping of the value π corresponding to the quick return.

327. On the other hand, when $a > 1$, the point B falls within the circle, and the value of y is represented by equation (2). The bar BP now makes complete revolutions, and when a is near to unity, swings very rapidly through a half-revolution in the positive direction. Hence, when the limiting case is reached by diminishing a to unity, the value of y increases uniformly at half the rate of x, but receives a sudden *increase* of π in value whenever x passes through an odd multiple of π.

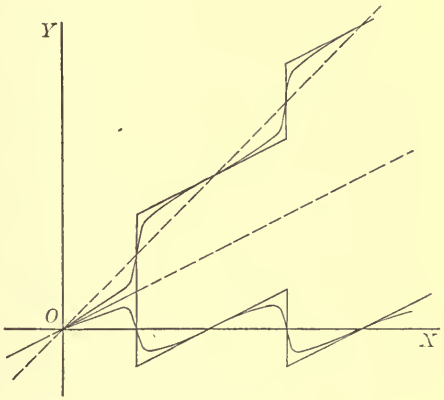

FIG. 61.

The graphs of the function y for two reciprocal values of a, near to unity, are given in Fig. 61. The zigzag lines, bisected respectively by $y=0$ and $y=x$, are the limiting forms to which the graph approaches as a approaches unity from the one side or the other.

Differentiation of Multiple-angle Series.

328. The differentiation of a sine-series or a cosine-series for $f(x)$ gives a cosine-series or a sine-series for $f'(x)$. When the series is an ordinary continuous function, and therefore either

an uneven or an even one, this is in accordance with the fact that the derivative of an uneven function is an even one, and *vice versa*. Moreover, the sine-series has the property of the uneven function that $f(0)=0$.

But when the series is the Fourier sine-series for an arbitrary function $f(x)$, we have *not* generally $f(0)=0$, although the sine-series as assumed in Art. 316 vanishes with x. This fact imposes discontinuity upon the series, $f(0)$ being *not* the value of the series when $x=0$, but the limit to which the value of the series approaches when x approaches to zero from the positive side.

329. The derivative of the Fourier cosine-series for $f(x)$ will in fact give the sine-series for $f'(x)$, true between the same limits. But it is to be noticed that the derivative of the sine-series for $f(x)$ cannot contain the term C, independent of x; therefore it cannot give the cosine-series for $f'(x)$ when the latter properly contains this term. The failure of differentiation in these cases is due to the fact that a divergent series is produced.

For example, putting $l=\pi$ in equation (1), Art. 320,

$$x=2(\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x-\dots), \quad \dots \quad (1)$$

of which the derivative is

$$1=2(\cos x-\cos 2x+\cos 3x+\dots),$$

which is inadmissible, because the series is divergent.

330. Let us now, in the general theorem, put $l=\pi$ (which is the same thing as putting x in place of its multiple $\pi x/l$, and therefore involves no loss of generality). Then, in the cosine-series for $f(x)$, Art. 317, the coefficient of $\cos nx$ is

$$A_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

Integrating by parts, this becomes

$$A_n=\frac{2}{\pi}\left[\frac{f(x)\sin nx}{n}-\frac{1}{n}\int f'(x)\sin nx\,dx\right]_0^\pi$$

$$=-\frac{2}{\pi n}\int_0^\pi f'(x)\sin nx\,dx=-\frac{1}{n}B_n',$$

where B_n' stands for the coefficient of sin nx in the sine-series for $f'(x)$ as found by the method of Art. 319. Thus $B_n'=-nA_n$; but this is precisely the coefficient of sin nx in the derivative of the cosine-series for $f(x)$. Thus the derivative series is the same as the result of developing $f'(x)$ in a Fourier sine-series by the general theorem.

331. On the other hand, the coefficient in the sine-series is

$$B_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx,$$

and, integrating by parts,

$$B_n=\frac{2}{\pi}\left[\frac{-f(x)\cos nx}{n}+\frac{1}{n}\int f'(x)\cos nx\,dx\right]_0^\pi$$

$$=\frac{2}{\pi}\left(\frac{f(0)}{n}-(-1)^n\frac{f(\pi)}{n}\right)+\frac{1}{n}A_n',$$

which shows that we shall have $A_n'=nB_n$, *only in case* $f(0)=0$ *and* $f(\pi)=0$. In this case only, therefore, and not in general,*

* The expression for B_n above shows that the sine-series for $f(x)$ can be separated into three parts as follows:

$$f(x)=A_1'\sin x+\tfrac{1}{2}A_2'\sin 2x+\tfrac{1}{3}A_3'\sin 3x+\ldots$$
$$+\frac{2f(0)}{\pi}[\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots]$$
$$+\frac{2f(\pi)}{\pi}[\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x-\ldots].$$

will the sine-series admit of a convergent derivative. It is to be noticed that this is precisely the case in which the equation of $f(x)$ to the sine-series *is true for the limits inclusive*, in which case the graph of the series is a continuous curve composed of arcs which meet each other with a common tangent. For example, $x(\pi-x)$ is such a function, and the corresponding sine-series (with a graph consisting of connected parabolic arcs) will be found to give a convergent cosine-series for the derivative $\pi-2x$. The result is, in fact, equivalent to equation (1), Art. 318.

Integration of Multiple-angle Series.

332. Conversely, if $f'(x)$ is a given function of which the development in a cosine-series contains no absolute term, that is, if $C'=0$, we shall have $\int_0^\pi f'(x)dx=0$, whence $f(\pi)=f(0)$. The constant of integration can be so taken that the integral $f(x)$ shall vanish when $x=0$, and we shall then have also $f(\pi)=0$.

The sum of the series in the third line is, by equation (1), Art. 329, equal to $\frac{1}{2}x$, and that in the second line is the result of substituting $\pi-x$ for x in the same equation; hence the last two lines are equivalent to

$$\frac{f(0)}{\pi}(\pi-x)+\frac{f(\pi)}{\pi}x=f(0)+\frac{f(\pi)-f(0)}{\pi}x,$$

and we have

$$f(x)=f(0)+\frac{f(\pi)-f(0)}{\pi}x+A_1'\sin x+\tfrac{1}{2}A_2'\sin 2x+\tfrac{1}{3}A_3'\sin 3x+\ldots. \qquad (1)$$

Since $C'=\frac{1}{\pi}\int_0^\pi f'(x)\,dx=\frac{f(\pi)-f(0)}{\pi}$, the derivative of this equation is

$$f'(x)=C'+A_1'\cos x+A_2'\cos 2x+A_3'\cos 3x+\ldots,$$

as given directly by Fourier's method. Thus in equation (1) the part of the sine-series which produces a divergent derivative has been already evaluated and the remainder admits of a convergent derivative.

The sine-series corresponding to $f(x)$ with the constant thus determined will, as shown above, have the property of actually approaching zero when $x=0$, and also when $x=\pi$.

In the case of any given value of $f'(x)$, we obtain a function of the required kind by simply transposing C' in the development in cosines. For example, from equation (1), Art. 318, we have

$$2x-\pi=-\frac{8}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right); \quad . \quad (1)$$

whence, integrating,

$$\pi x-x^2=\frac{8}{\pi}\left(\sin x+\frac{1}{3^3}\sin 3x+\frac{1}{5^3}\sin 5x+\ldots\right), \quad . \quad (2)$$

the constant of integration being zero, so that the first member vanishes when $x=0$, and it is found to vanish also when $x=\pi$.

333. In integrating a sine-series the direct determination of the constant is not so readily made, but by proceeding to the next integration we can determine both constants by the double condition employed above. For example, integrating the equation obtained above, and multiplying by -3, we have

$$x^3-\frac{3\pi}{2}x^2+C=\frac{24}{\pi}\left(\cos x+\frac{1}{3^4}\cos 3x+\frac{1}{5^4}\cos 5x+\ldots\right);$$

and integrating again,

$$\frac{x^4}{4}-\frac{\pi}{2}x^3+Cx+C'=\frac{24}{\pi}\left(\sin x+\frac{1}{3^5}\sin 3x+\frac{1}{5^5}\sin 5x+\ldots\right),$$

in which, the two conditions that the first member must vanish when $x=0$ and also when $x=\pi$ give $C'=0$ and $C=\frac{1}{4}\pi^3$.

Hence the successive integrals are

$$x^3-\frac{3\pi}{2}x^2+\frac{\pi^3}{4}=\frac{24}{\pi}\left(\cos x+\frac{1}{3^4}\cos 3x+\frac{1}{5^4}\cos 5x+\ldots\right), \quad (3)$$

$$x^4-2\pi x^3+\pi^3 x=\frac{96}{\pi}\left(\sin x+\frac{1}{3^5}\sin 3x+\frac{1}{5^5}\sin 5x+\ldots\right). \quad (4)$$

334. We have seen that the sine-series obtained directly for a given function is in general discontinuous, and represents the function for values between but *exclusive of the limiting values* of the variable. But the preceding articles show that the expression for the function found by integrating the derivative as a cosine-series gives its true value *inclusive* of the limits. The value of the sine-series thus found differs from the given function by a certain linear function.

As an illustration, if we develop e^{mx} in a sine-series for values of x between 0 and π (see the formulæ of Art. 323), we have

$$B_n=\frac{2}{\pi}\int_0^\pi e^{mx}\sin nx\,dx=\frac{2n}{\pi(m^2+n^2)}[1-(-1)^n e^{m\pi}],$$

giving the development

$$e^{mx}=\frac{2}{\pi}\left(\frac{1+e^{m\pi}}{m^2+1^2}\sin x+2\frac{1-e^{m\pi}}{m^2+2^2}\sin 2x+3\frac{1+e^{m\pi}}{m^2+3^2}\sin 3x+\ldots\right). \quad (1)$$

Again, developing e^{mx} in a cosine-series, we have

$$C=\frac{1}{\pi}\int_0^\pi e^{mx}dx=\frac{e^{m\pi}-1}{m\pi}$$

and

$$A_n=\frac{2}{\pi}\int_0^\pi e^{mx}\cos nx\,dx=\frac{2m}{\pi(m^2+n^2)}[(-1)^n e^{m\pi}-1],$$

giving the development

$$e^{mx} = \frac{e^{m\pi}-1}{m\pi} - \frac{2m}{\pi}\left(\frac{1+e^{m\pi}}{m^2+1^2}\cos x + \frac{1-e^{m\pi}}{m^2+2^2}\cos 2x + \frac{1+e^{m\pi}}{m^2+3^2}\cos 3x + \dots\right) \quad (2)$$

Integrating and multiplying by m,

$$e^{mx} = \frac{e^{m\pi}-1}{\pi}x + 1 - \frac{2m^2}{\pi}\left(\frac{1+e^{m\pi}}{m^2+1^2}\sin x + \frac{1-e^{m\pi}}{2(m^2+2^2)}\sin 2x + \dots\right), \quad (3)$$

the constant of integration being determined by either of the conditions that the series vanishes when $x=0$ and when $x=\pi$.

335. The coefficient of $\sin nx$ in the last series is

$$-\frac{2m^2}{\pi}\,\frac{1-(-1)^n e^{m\pi}}{n(m^2+n^2)},$$

and subtracting this from B_n, the coefficient in equation (1), we have

$$\frac{2[1-(-1)^n e^{m\pi}]}{\pi n(m^2+n^2)}(n^2+m^2) = \frac{2}{n\pi}[1-(-1)^n e^{m\pi}];$$

therefore subtracting equation (3) from equation (1) we find

$$\frac{e^{m\pi}-1}{\pi}x + 1 = \frac{2}{\pi}\sum_{n=1}^{n=\infty}\frac{1-(-1)^n e^{m\pi}}{n}\sin nx.$$

This equation is readily verified when we substitute in the first member the values of x and of unity in the form of sine-series, viz.

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \dots), \quad . \quad . \quad . \quad (1)$$

see Art. 329, and

$$1 = \frac{4}{\pi}[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots], \quad . \quad . \quad . \quad (2)$$

see Art. 336.

Thus the linear function in equation (3) forms, as it were, that part of the function e^{mx} which requires for its development in sines a discontinuous series. When the complete graph of the second member of equation (3) is drawn, the linear function represents the straight line which passes through the extremities of the arc of $y=e^{mx}$ with which the graph coincides between $x=0$ and $x=\pi$.

Series obtained by Transformation.

336. Denoting by x the angle whose multiples appear in the series (which was denoted by θ in Art. 316), the limits of applicability of the series derived directly by the methods of Arts. 317 and 319 are $x=0$ and $x=\pi$; but by transformation of such series, others with different limits may be derived. We notice in the first place, however, that the substitution of $\pi-x$ for x does not change the limits. Its effect, moreover, is merely to change the sign of alternate terms, namely those containing odd multiples of x in the case of a cosine-series, and those containing even multiples in the case of a sine-series. It follows that the sum of the given series and this transformation, and likewise their difference, will contain either odd multiples only or even multiples only of x.

For example, equation (1) of the preceding article gives

$$\pi-x=2(\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\dots), \quad . \quad . \quad . \quad (3)$$

and the sum is

$$\pi=4(\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\dots), \quad . \quad . \quad . \quad (4)$$

which is equation (2) employed above. Again, the difference of the series for x and for $\pi-x$ is

$$\pi-2x=4(\tfrac{1}{2}\sin 2x+\tfrac{1}{4}\sin 4x+\tfrac{1}{6}\sin 6x+\dots), \quad . \quad . \quad (5)$$

which contains only even multiples of x.

337. A series containing even multiples only of x may be transformed by putting $\phi = 2x$; and then, since the limits for x are 0 and π, those for ϕ are 0 and 2π. For example, equation (5) above thus becomes

$$\pi - \phi = 2(\sin\phi + \tfrac{1}{2}\sin 2\phi + \tfrac{1}{3}\sin 3\phi + \dots),$$

which is identical with equation (3), and shows that that equation is true up to $x = 2\pi$. This might also have been inferred from the fact that equation (1) is true between the limits $-\pi$ and π.

If a multiple-angle series whose limits are 0 and π be transformed by putting $\frac{1}{2}\pi - x$ for x, the limits for the new variable are $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. In general, the transformed series would contain both sines and cosines; but, if odd multiples only occur, a sine-series will thus give rise to a cosine-series, and *vice versa*. For example, equation (4) above becomes

$$\frac{\pi}{4} = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \dots,$$

which is true for values of x between the limits $\pm\frac{1}{2}\pi$, *exclusive* of both limits.

Functions with Arbitrary Discontinuities.

338. The restriction of the validity of equations (4), Art. 317, and (3), Art. 319, to values of x between 0 and l presents itself as a natural consequence of the fact that the definite integrals in the second member *depend for their values solely upon the values of the function for values of x between the limits of integration.*

In fact, it is found that the equations are true when the values of the function for values of x between the limits are defined in

any manner whatever, provided only there is no ambiguity about the values of the definite integrals. Thus $f(x)$ may be defined by one functional expression, say $f_1(x)$, from $x=0$ to $x=a$, by another expression, $f_2(x)$, from $x=a$ to $x=b$; and so on to $x=l$. It is not even necessary that the values of $f_1(a)$ and $f_2(a)$ shall be the same. In other words, the graph of $y=f(x)$ may consist, as in Fig. 62, of any disconnected arcs, of which the projections upon the axis of x cover without overlapping the range from $x=0$ to $x=l$. The values of the definite integrals are now found by separate integrations of the functions $f_1(x)$, $f_2(x)$ etc., each over its proper range. In particular, that occurring in the value of C, equation (3), Art. 317, namely

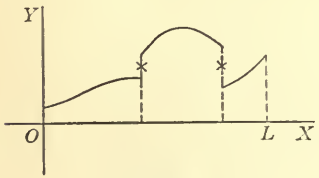

FIG. 62.

$$C=\frac{1}{l}\int_0^l f(x)dx,$$

represents the total area between the axis of x, the arcs, and their ordinates, due regard being paid to algebraic sign. Thus the absolute term is the mean value of the ordinate between $x=0$ and $x=l$.

It is a noteworthy fact, which we here state without proof, that the value of either series corresponding to a value $x=a$ at which discontinuity occurs is $\frac{1}{2}\{f_1(a)+f_2(a)\}$, that is, the arithmetical mean between the two values of the function.

339. As an illustration, let us construct a cosine-series in which $f_1(x)=x$ from $x=0$ to $x=\frac{1}{2}l$, and $f_2(x)=0$ for all values of x between $\frac{1}{2}l$ and l. We shall now have in equations (3) and (2), Art. 317, $C=\frac{1}{8}l$, and

$$A_n=\frac{2}{l}\int_0^{\frac{1}{2}l} x\cos n\frac{\pi x}{l}dx=\frac{2l}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\cos n\theta\,d\theta=\frac{2l}{n^2\pi^2}\int_0^{\frac{n\pi}{2}}\phi\cos\phi\,d\phi.$$

Integrating by parts,

$$\int_0^{\frac{n\pi}{2}} \phi \cos\phi \, d\phi = \phi \sin\phi \Big]_0^{\frac{n\pi}{2}} - \int_0^{\frac{n\pi}{2}} \sin\phi \, d\phi$$

$$= n\frac{\pi}{2} \sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - 1.$$

The successive values of this integral for $n = 1, 2, 3$, etc., are

$$\frac{\pi}{2} - 1, \quad -2, \quad -\frac{3\pi}{2} - 1, \quad 0, \quad \frac{5\pi}{2} - 1, \quad -2, \text{ etc.}$$

Hence, substituting in the general equation, we have

$$f(x) = \frac{l}{8} + \frac{l}{\pi}\left(\cos\frac{\pi x}{l} - \frac{1}{3}\cos 3\frac{\pi x}{l} + \frac{1}{5}\cos 5\frac{\pi x}{l} - \ldots\right)$$

$$- \frac{2l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos 3\frac{\pi x}{l} + \frac{1}{5^2}\cos 5\frac{\pi x}{l} + \ldots\right)$$

$$- \frac{l}{\pi^2}\left(\cos 2\frac{\pi x}{l} + \frac{1}{3^2}\cos 6\frac{\pi x}{l} + \frac{1}{5^2}\cos 10\frac{\pi x}{l} + \ldots\right).$$

It is readily verified, by means of Gregory's series for $\frac{1}{4}\pi$ and the series quoted in Art. 318, that $x = 0$ and $x = l$ each give to the second member the value zero. Again, if we put $x = \frac{1}{2}l$, the second member becomes

$$\frac{l}{8} + \frac{l}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) = \frac{l}{4},$$

and this is the arithmetical mean between $f_1(\frac{1}{2}l) = \frac{1}{2}l$ and $f_2(\frac{1}{2}l) = 0$, in accordance with the last paragraph of Art. 338.

Formulæ Involving both Sines and Cosines.

340. If, in the half-sum of the formulæ of Arts. 317 and 319, the factors $\sin n\pi x/l$ and $\cos n\pi x/l$ be placed under the integral signs, the result may be written

$$f(x)=\frac{1}{2l}\int_0^l f(v)\,dv+\frac{1}{l}\sum_{n=1}^{n=\infty}\int_0^l f(v)\cos n\frac{\pi}{l}(v-x)\,dv. \quad . \quad (1)$$

In like manner, the formula of Art. 322 may be written

$$f(x)=\frac{1}{2l}\int_{-l}^l f(v)\,dv+\frac{1}{l}\sum_{n=1}^{n=\infty}\int_{-l}^l f(v)\cos n\frac{\pi}{l}(v-x)\,dv.* \quad . \quad (2)$$

Let $u_1=\pi/l$, $u_2=2\pi/l$, etc., so that $u_n=n\pi/l$ and the series of u's have the constant difference $\Delta u=\pi/l$. Equation (2) may then be written

$$f(x)=\frac{1}{2l}\int_{-l}^l f(v)\,dv+\frac{1}{\pi}\sum_{n=1}^{n=\infty}\int_{-l}^l \cos u_n(v-x)f(v)\,dv\,\Delta u.$$

If, in this equation, l increases without limit, so that Δu decreases without limit, the summation with respect to n will be replaced by integration, the limits for u being zero and infinity. Therefore, if $f(x)$ is a function such that $\int_{-\infty}^{\infty} f(v)dv$ has a finite value, we shall have

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \cos u(v-x)f(v)dv\,du.$$

This is known as Fourier's *double-integral theorem.*

* Poisson's method consists in a direct demonstration of this equation. See Todhunter's Int. Calc., p. 298.

Examples XXI.

1. Develop x^2 in a cosine-series, taking $l=\pi$; and verify for $x=0$ and for $x=\pi$.

$$x^2=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\right).$$

2. Find the sine-series for $f(x)=1$ from $x=0$ to $x=l$, and show that the result is the derivative of equation (1), Art. 318.

$$1=\frac{4}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin 3\frac{\pi x}{l}+\frac{1}{5}\sin 5\frac{\pi x}{l}+\ldots\right).$$

3. Develop $x(l-x)$ in a series of sines.

$$x(l-x)=\frac{8l^2}{\pi^3}\left(\sin\frac{\pi x}{l}+\frac{1}{3^3}\sin 3\frac{\pi x}{l}+\frac{1}{5^3}\sin 5\frac{\pi x}{l}+\ldots\right).$$

4. Expand $\cos\frac{1}{2}\theta$ in a series of cosines taking $l=\pi$, and show that the result is numerically true for all values of θ.

$$\cos\tfrac{1}{2}\theta=\frac{4}{\pi}\left(\frac{1}{2}+\frac{\cos\theta}{1\cdot 3}-\frac{\cos 2\theta}{3\cdot 5}+\frac{\cos 3\theta}{5\cdot 7}-\ldots\right).$$

5. Expand $\cos\frac{1}{2}\theta$ in a series of sines, and show by its graph that the result is true from $\theta=0$ to $\theta=2\pi$ exclusive of these limiting values.

$$\cos\tfrac{1}{2}\theta=\frac{8}{\pi}\left(\frac{\sin\theta}{1\cdot 3}+\frac{2\sin 2\theta}{3\cdot 5}+\frac{3\sin 3\theta}{5\cdot 7}+\ldots\right).$$

6. From Example 4 derive the numerical series

$$\frac{\pi}{8}=\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\ldots,$$

and verify by means of Gregory's series.

7. Show that the result in Example 1 may be derived by integrating the expression for x in sines, and determine the constant together with the result of the next integration.

$$x^3-\pi^2x=-12\left(\sin x-\frac{1}{2^3}\sin 2x+\frac{1}{3^3}\sin 3x-\ldots\right)$$

8. Find the result of two more successive integrations, see Example 7.

$$x^4-2\pi^2x^2+\frac{7}{15}\pi^4=48\left(\cos x-\frac{1}{2^4}\cos 2x+\frac{1}{3^4}\cos 3x-\ldots\right),$$

$$3x^5-10\pi^2x^3+7\pi^4x=720\left(\sin x-\frac{1}{2^5}\sin 2x+\frac{1}{3^5}\sin 3x-\ldots\right).$$

9. Putting $S_n=1+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$ as in Diff. Calc., Art. 239, we have also $1-\frac{1}{2^n}+\frac{1}{3^n}-\ldots=\left(1-\frac{1}{2^{n-1}}\right)S_n$. Derive, by putting $x=\pi$ and $x=0$ in Examples 1 and 8, the values of S_2 and S_4.

In like manner all the values of S_{2n} might be found and thence the Bernoullian numbers. The series of integrations commenced in Arts. 332 and 333 gives another method of calculating S_{2n}, since

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\ldots=\left(1-\frac{1}{2^n}\right)S_n.$$

10. From the cosine-series for cosh mx (taking $l=\pi$), Art. 323, derive

$$\sinh mx=\frac{\sinh m\pi}{\pi}x$$
$$-\frac{2m^2\sinh m\pi}{\pi}\left(\frac{\sin x}{m^2+1}-\frac{\sin 2x}{2(m^2+2^2)}+\frac{\sin 3x}{3(m^2+3^2)}-\ldots\right),$$

and show that this result agrees with the value of sinh mx in Art. 323.

11. Trace the graph of the quasi-periodic function

$$y=(1-a^2)\int_0^x\frac{dx}{1+2a\cos x+a^2},\qquad a<1.$$

See equation (1), Art. 312, showing that one of the points farthest from the bisecting line $y=x$ is on the line $y=\pi-x$. Show also by formula (G), p. 41, that

$$y=2\tan^{-1}\left(\frac{1-a}{1+a}\tan\tfrac{1}{2}x\right),$$

and discuss the case in which $a=1$.

12. Derive the expansion

$$\log \sin \tfrac{1}{2}x = -\log 2 - (\cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \ldots);$$

and thence the value of $\int_0^\pi \log \sin \tfrac{1}{2}x \cdot \cos nx\, dx$. $\quad -\frac{\pi}{2n}$.

13. Evaluate $\int_0^\pi \tan^{-1}\frac{a \sin x}{1 - a\cos x} \sin nx\, dx$.

See Art. 311. $\quad \frac{\pi a^n}{2n}$.

14. Derive the expansion, when $a<1$,

$$\int_0^x \tan^{-1}\frac{a \sin x\, dx}{1 + a\cos x} = a - \frac{a^2}{2^2} + \frac{a^3}{3^2} - \ldots - \left(a\cos x - \frac{a^2}{2^2}\cos 2x + \frac{a^3}{3^2}\cos 3x - \ldots\right),$$

thence, putting $a=1$, derive the result of Example 1, and show why it is limited to values between $-\pi$ and π.

15. Show that for all values of x between $\pm\frac{1}{2}\pi$ inclusive of these limiting values

$$x = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right),$$

and trace the complete graph of the second member.

16. By means of the sine-series for $x(\pi - x)$ show that

$$x^2 = \frac{\pi^2}{4} - \frac{8}{\pi}\left(\cos x - \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} - \ldots\right)$$

for all values of x from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ inclusive of the limits, and thence derive the numerical series. $\quad 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{3^2}$.

Compare Diff. Calc. (*Ex. XXIII*, 22). *This is also the result of putting* $x=\frac{1}{2}\pi$ *in Example* 7, *and the others of the same set result from the sine-series for higher polynomials, see Example* 8.

17. From the result of Example 4, by means of transformations and the trigonometric formula

$$\cos \tfrac{1}{2}\theta - \sin \tfrac{1}{2}\theta = \sqrt{2} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right),$$

show that, for values of θ from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ inclusive,

$$\sin \tfrac{1}{2}\theta = \frac{4\sqrt{2}}{\pi}\left(\frac{\sin\theta}{1\cdot 3} - \frac{\sin 3\theta}{5\cdot 7} + \frac{\sin 5\theta}{9\cdot 11} - \dots\right);$$

also that the same series represents $\cos \frac{1}{2}\theta$ from $\theta = \frac{1}{2}\pi$ to $\theta = \frac{3}{2}\pi$.

18. In a similar manner show that

$$\cos \tfrac{1}{2}\theta = \frac{4\sqrt{2}}{\pi}\left(\frac{1}{2} + \frac{\cos 2\theta}{3\cdot 5} - \frac{\cos 4\theta}{7\cdot 9} + \dots\right)$$

when θ is between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and that this series always represents the greater of the two quantities $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ taken positively.

The equations derived in like manner from Example 5, *or by taking derivatives of those above are, for values between* $\pm\frac{1}{2}\pi$,

$$\sin \tfrac{1}{2}\theta = \frac{8\sqrt{2}}{\pi}\left(\frac{2\sin 2\theta}{3\cdot 5} - \frac{4\sin 4\theta}{7\cdot 9} + \frac{6\sin 6\theta}{11\cdot 13} - \dots\right),$$

$$\cos \tfrac{1}{2}\theta = \frac{8\sqrt{2}}{\pi}\left(\frac{\cos\theta}{1\cdot 3} - \frac{3\cos 3\theta}{5\cdot 7} + \frac{5\cos 5\theta}{9\cdot 11} - \dots\right),$$

in the first of which the limits are excluded.

19. Expand $\cos mx$ in cosines of multiples of x, and verify the result in the cases where m becomes 0 or an integer.

$$\cos mx = \frac{\sin m\pi}{m\pi}\left[1 + 2m^2\left(\frac{\cos x}{1^2 - m^2} - \frac{\cos 2x}{2^2 - m^2} + \frac{\cos 3x}{3^2 - m^2} - \dots\right)\right].$$

20. Expand $\sin mx$ in a cosine-series for values of x between 0 and π.

$$\sin mx = \frac{1 - \cos m\pi}{m\pi}\left[1 - 2m^2\left(\frac{\cos 2x}{2^2 - m^2} + \frac{\cos 4x}{4^2 - m^2} + \frac{\cos 6x}{6^2 - m^2} + \dots\right)\right]$$
$$- 2m\frac{1 + \cos m\pi}{\pi}\left(\frac{\cos x}{1^2 - m^2} + \frac{\cos 3x}{3^2 - m^2} + \frac{\cos 5x}{5^2 - m^2} + \dots\right).$$

When m is an integer, one line (inclusive of the term which takes the indeterminate form) vanishes.

21. Expand $\cos mx$ and $\sin mx$ in sine-series.

$$\cos mx = 2\frac{1+\cos m\pi}{\pi}\left(\frac{\sin x}{1^2-m^2}+\frac{3\sin 3x}{3^2-m^2}+\frac{5\sin 5x}{5^2-m^2}+\dots\right)$$
$$+2\frac{1-\cos m\pi}{\pi}\left(\frac{2\sin 2x}{2^2-m^2}+\frac{4\sin 4x}{4^2-m^2}+\dots\right)$$
$$\sin mx = \frac{2\sin m\pi}{\pi}\left(\frac{\sin x}{1^2-m^2}-\frac{2\sin 2x}{2^2-m^2}+\frac{3\sin 3x}{3^2-m^2}-\dots\right).$$

22. Find the sine-series for $\cos 3x$, and thence show that from $x=0$ to $x=2\pi$

$$\cos\frac{3}{2}x=\frac{8}{\pi}\left(-\frac{\sin x}{5}+\frac{2\sin 2x}{1.7}+\frac{3\sin 3x}{3.9}+\dots\right).$$

23. If for values between 0 and π inclusive

$$f(x)=C+\sum_{n=1}^{n=\infty}A_n\cos nx \quad\text{and}\quad \phi(x)=C'+\sum_{n=1}^{n=\infty}A'_n\cos nx,$$

prove that

$$\int_0^\pi f(x)\phi(x)dx=\frac{\pi}{2}\left(2CC'+\sum_{n=1}^{n=\infty}A_nA'_n\right).$$

24. Show by means of Example 23 and equation (5), Art. 311, that, when $a<1$,

$$\int_0^\pi[\log(1-2a\cos x+a^2)]^2dx=2\pi\left(a^2+\frac{a^4}{2^2}+\frac{a^6}{3^2}+\dots\right).$$

25. Prove that

$$\int_0^\pi\frac{dx}{(1-2a\cos x+a^2)^2}=\pi\frac{1+a^2}{(1-a^2)^3}.$$

26. Prove that

$$\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\dots=\frac{\pi^2-8}{16}.$$

27. Prove that

$$\int_0^{\frac{\pi}{2}} (\log \cos \phi)^2 \, d\phi = \int_0^{\frac{\pi}{2}} (\log \sin \phi)^2 \, d\phi = \frac{\pi(\log 2)^2}{2} + \frac{\pi^3}{24}$$

(see Art. 321 and Ex. 12); also that

$$\int_0^{\frac{\pi}{2}} \log \cos \phi \cdot \log \sin \phi \, d\phi = \frac{\pi(\log 2)^2}{2} - \frac{\pi^3}{48}.$$

28. Putting $\phi(x) = \dfrac{1-a^2}{1-2a\cos x+a^2}$ in the Theorem of Example 23, prove that, when $a<1$,

$$(1-a^2)\int_0^{\pi} \frac{f(x)dx}{1-2a\cos x+a^2} = \pi(C+A_1a+A_2a^2+\dots);$$

and thence that, when a approaches unity, the limiting value of the first member is $\pi f(0)$.

29. Prove that

$$\int_0^{\pi} \frac{\log(1-2b\cos x+b^2)}{1-2a\cos x+a^2}\,dx = \frac{2\pi \log(1-ab)}{1-a^2}.$$

30. If for values of x between 0 and π

$$f(x) = \sum_{n=1}^{n=\infty} B_n \sin nx \quad \text{and} \quad \phi(x) = \sum_{n=1}^{n=\infty} B'_n \sin nx,$$

prove that

$$\int_0^{\pi} f(x)\,\phi(x)dx = \frac{\pi}{2}\sum_{n=1}^{n=\infty} B_nB'_n.$$

31. Prove that

$$\int_0^{\pi} f(x)\,dx = 2\left(B_1+\frac{B_3}{3}+\frac{B_5}{5}+\dots\right)$$

by direct integration and also as a case of the Theorem of Example 30.

32. Evaluate $\int_0^\pi \frac{\sin^2 x\,dx}{(1+2a\cos x+a^2)^2}$, when $a<1$, and when $a>1$.

$$\frac{\pi}{2(1-a^2)};\quad \frac{\pi}{2a^2(a^2-1)}.$$

33. By means of Examples 23 and 30 and the trigonometric formula $\cos 2mx=\cos^2 mx-\sin^2 mx$, derive the result

$$\frac{\pi\cot m\pi}{2m}=\frac{1}{2m^2}-\frac{1}{1^2-m^2}-\frac{1}{2^2-m^2}-\frac{1}{3^2-m^2}-\cdots$$

34. Prove in like manner

$$\frac{\pi\coth m\pi}{2m}=\frac{1}{2m^2}+\frac{1}{m^2+1^2}+\frac{1}{m^2+2^2}+\cdots$$

35. Show that

$$\int_0^{\frac{1}{2}\pi} x^2\log\cos x\,dx=-\frac{\pi^3\log 2}{24}-\frac{\pi}{4}S_3,$$

$$\int_0^{\frac{1}{2}\pi} x^2\log\sin x\,dx=-\frac{\pi^3\log 2}{24}+\frac{3\pi}{16}S_3,$$

$$\int_0^{\pi} x^2\log\sin x\,dx=-\frac{\pi^3\log 2}{3}-\frac{\pi}{2}S_3,$$

where $S_3=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots$

36. Construct a cosine-series whose value shall be $+1$ from $x=0$ to $x=\alpha$, zero from $x=\alpha$ to $x=\pi-\alpha$, and -1 from $x=\pi-\alpha$ to $x=\pi$. Verify the statement in Art. 338 with regard to the value of $f(\alpha)$.

$$f(x)=\frac{4}{\pi}[\sin\alpha\cos x+\frac{1}{3}\sin 3\alpha\cos 3x+\frac{1}{5}\sin 5\alpha\cos 5x+\cdots].$$

37. Show that the integral of the series in Example 36, namely

$$\frac{4}{\pi}\left(\sin\alpha\sin x+\frac{1}{3^2}\sin 3\alpha\sin 3x+\frac{1}{5^2}\sin 5\alpha\sin 5x+\cdots\right),$$

represents x from $x=0$ to $x=\alpha$, α from $x=\alpha$ to $x=\pi-\alpha$, and $\pi-x$ from $x=\pi-\alpha$ to $x=\pi$.

XXII.

The Eulerian Integrals.

341. A definite integral with constant limits, while not a function of the variable whose differential appears under the integral sign, is in general a function of any other algebraic quantity which occurs in its expression; and, if that quantity admits of continuous variation, the integral is a continuous function of it, and admits of differentiation with respect to it. For example, in the integral $\int_0^1 x^n dx$, n admits of any value greater than -1; therefore the integral is, for the range of values from -1 to $+\infty$, a continuous function of n. In this case, the value of the integral as a function of n and of its derivatives with respect to n are readily expressed by means of the elementary functions. See Art. 278.

Certain ones among those definite integrals which are *not* thus "integrable" in elementary functions have come to be regarded as fundamental ones, and serve to define new functions which are employed in the expression and calculation of other integrals of more complex form. Of these the most important are those known as the *First* and *Second Eulerian Integrals.*

342. The first Eulerian Integral is a function of two variables denoted by $B(m, n)$, and hence sometimes called *the Beta Function.* It is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,$$

in which each of the exponents must be greater than -1; therefore m and n are restricted to *positive values.* By Art. 97, $B(n, m) = B(m, n)$, so that m and n are interchangeable. The

value of the integral, when one or both of the variables, m and n, is an integer, is expressible in elementary functions; it may be found by a formula of reduction, or directly as follows:

Let n have any positive value, then

$$\int_0^1 x^{m-1}dx = \frac{1}{m} \quad \text{and} \quad \int_0^1 x^m dx = \frac{1}{m+1}.$$

Subtracting,

$$\int_0^1 x^{m-1}(1-x)dx = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}. \quad \ldots \quad (1)$$

In like manner, putting $m+1$ in place of m,

$$\int_0^1 x^m(1-x)dx = \frac{1}{(m+1)(m+2)};$$

and subtracting this from equation (1),

$$\int_0^1 x^{m-1}(1-x)^2 = \frac{1}{m+1}\left[\frac{1}{m} - \frac{1}{m+2}\right] = \frac{2}{m(m+1)(m+2)}. \quad (2)$$

Again, putting $m+1$ in place of m,

$$\int_0^1 x^m(1-x)^2dx = \frac{2}{(m+1)(m+2)(m+3)};$$

and subtracting,

$$\int_0^1 x^{m-1}(1-x)^3dx = \frac{2}{(m+1)(m+2)}\left[\frac{1}{m} - \frac{1}{m+3}\right]$$

$$= \frac{2.3}{m(m+1)(m+2)(m+3)}. \quad \ldots \quad (3)$$

It is evident that, in this manner, we can prove

$$\int_0^1 x^{m-1}(1-x)^n dx = \frac{n!}{m(m+1)\ldots(m+n)} \quad . \quad . \quad . \quad (4)$$

for all integral values of n. This equation gives the value of $B(m, n+1)$ when n is an integer.

Gauss' II Function.

343. When m is fixed and n increases without limit, $B(m, n)$ approaches zero as its limit; but its ratio to n^{-m} is found to be finite. This ratio constitutes a function of the single variable m known as the Second Eulerian Integral.

The limiting ratio in question may evidently be derived from equation (4) above by making the integer n increase without limit. First transforming the integral by putting $x=y/n$, the equation becomes

$$\frac{1}{n^m}\int_0^n y^{m-1}\left(1-\frac{y}{n}\right)^n dy = \frac{1}{m}\,\frac{1}{1+m}\,\frac{2}{2+m}\,\frac{3}{3+m}\cdots\frac{n}{n+m}.$$

When n is made infinite, we have, by evaluation of an indeterminate form,

$$\left[\left(1-\frac{y}{n}\right)^n\right]_{n=\infty} = e^{-y}.$$

Hence the equation may be written in the form

$$\int_0^\infty y^{m-1}e^{-y}dy = \frac{1}{m}\left[n^m \cdot \frac{1}{1+m}\,\frac{2}{2+m}\cdots\frac{n}{n+m}\right]_{n=\infty} \quad . \quad . \quad (1)$$

Here the first member has a finite value, while in the second member the factor n^m becomes infinite, and the continued product (in which each factor is a proper fraction) has zero for its

limit when the number of factors is infinite. The integral (in which m must be positive) is *the Second Eulerian Integral.*

344. Gauss denoted by the symbol Π the function defined by

$$\Pi(m)=\left[n^m\frac{1}{1+m}\,\frac{2}{2+m}\,\frac{3}{3+m}\cdots\frac{n}{n+m}\right]_{n=\infty}. \quad . \quad . \quad (1)$$

Thus the equation above may be written

$$\int_0^\infty y^{m-1}e^{-y}dy=\frac{\Pi(m)}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

He also defined $\Pi(-m)$ as the result of changing the sign of m in equation (1),* so that

$$\Pi(-m)=\left[n^{-m}\frac{1}{1-m}\,\frac{2}{2-m}\,\frac{3}{3-m}\cdots\frac{n}{n-m}\right]_{n=\infty}. \quad . \quad . \quad (3)$$

In this last expression, the factor n^{-m} vanishes, and the continued product becomes infinite, when n is infinite.

Now multiplying equations (1) and (3), we find

$$\Pi(m)\Pi(-m)=\left[\frac{1}{1-m^2}\,\frac{2^2}{2^2-m^2}\,\frac{3^2}{3^2-m^2}\cdots\frac{n^2}{n^2-m^2}\right]^{\dagger}_{n=\infty}. \quad (4)$$

* It is to be noticed, however, that *the integral* in equation (2) does not admit of negative values of m.

† The fractions in equations (1) and (3), taken together, form a series running to infinity in both directions. In equation (4), an equal number of factors is taken from each, and then the number is made infinite, the object being to get rid of the power of this number before it is made infinite. This could, however, be done by taking n factors from the series (3) and rn factors from the series (1), r being constant; for $(rn)^m\times n^{-m}$ reduces to r^m, and thus remains finite when n is made infinite. Thus the product $\Pi(m)\Pi(-m)$ may be written in the form

$$r^m\left[\frac{n}{n-m}\,\frac{n-1}{n-1-m}\cdots\frac{1}{1-m}\,\frac{1}{1+m}\,\frac{2}{2+m}\cdots\frac{rn}{rn+m}\right]_{n=\infty}.$$

This expression has therefore a value *independent of* r. Supposing $r>1$, the expression contains an infinite number of extra factors on the right. It follows that the product of these factors must be r^{-m}. See the note on p. 234 Diff. Calc., in which the series is the reciprocal of that here considered.

Putting $m=\phi/\pi$, the reciprocal of the second member becomes the continued product

$$\left[1-\frac{\phi^2}{\pi^2}\right]\left[1-\frac{\phi^2}{2^2\pi^2}\right]\left[1-\frac{\phi^2}{3^2\pi^2}\right]\cdots,$$

where the number of factors is infinite. The value of this is, by Diff. Calc., Art. 234, $\sin\phi/\phi$. Therefore

$$\Pi(m)\Pi(-m)=\frac{\phi}{\sin\phi}=\frac{m\pi}{\sin m\pi}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation expresses a fundamental property of the function Π, and shows that $\Pi(-m)$ is finite except when m is an integer.

345. If, in equation (1), we put $m+1$ in place of m, we have

$$\Pi(m+1)=n^{m+1}\frac{1}{2+m}\frac{2}{3+m}\cdots\frac{n}{n+1+m}\Bigg]_{n=\infty},$$

hence, comparing with equation (1), we have

$$\Pi(m+1)=\Pi(m)\frac{n(1+m)}{n+1+m}\Bigg]_{n=\infty}=(m+1)\Pi(m), \quad . \quad . \quad (6)$$

a second fundamental property of the function.

Putting $m=0$ in equation (1), we have $\Pi(0)=1$; whence, by equation (6), we derive successively, $\Pi(1)=1$, $\Pi(2)=2!$, and in general when p is a positive integer $\Pi(p)=p!$. Thus the function is a *generalization of the factorial product;* this is in fact directly evident on giving to m in equation (1) an integral value.

The Gamma Function.

346. The definite integral in equation (2), Art. 344, is *the Second Eulerian Integral*, and according to Legendre's notation is denoted by $\Gamma(m)$; it is therefore generally called *the Gamma Function*. Thus

$$\Gamma(m)=\int_0^\infty x^{m-1}e^{-x}dx; \quad \ldots\ldots \quad (1)$$

and, comparing with Gauss' notation,

$$\Pi(m)=m\Gamma(m). \quad \ldots\ldots \quad (2)$$

Substituting in equation (6) of the preceding article, we have the fundamental relation

$$\Gamma(m+1)=m\Gamma(m), \quad \ldots\ldots \quad (3)$$

which is, in fact, equivalent to the formula of reduction given in Art. 92. We have therefore also

$$\Pi(m)=\Gamma(m+1), \quad \ldots\ldots \quad (4)$$

so that a table of the values of the Γ-function is also a table of values of the Π-function.

Equation (2), together with the general definition of $\Pi(m)$, serves to define $\Gamma(m)$ for negative as well as positive values of m. Equation (3) may also be written in the form

$$\Gamma(n-1)=\frac{\Gamma(n)}{n-1}. \quad \ldots\ldots \quad (5)$$

Again, by equations (3) and (4), the fundamental property

expressed in equation (5), Art. 344, becomes

$$\Gamma(m)\Gamma(1-m)=\frac{\pi}{\sin m\pi}. \quad \ldots\ldots \quad (6)$$

347. Equation (1) gives at once $\Gamma(1)=1$; whence, by successive applications of equation (3), $\Gamma(p)=(p-1)!$, when p is a positive integer. By equation (5), $\Gamma(0)=\infty$, and $\Gamma(n)$ is also infinite for all negative integral values of n. Putting $m=\frac{1}{2}$ in equation (6), we derive $\Gamma\frac{1}{2})=\sqrt{\pi}$; whence, again using equations (3) and (5), $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2})=\frac{3}{4}\sqrt{\pi}$, etc., $\Gamma(-\frac{1}{2})=-2\sqrt{\pi}$, $\Gamma(-\frac{3}{2})=\frac{4}{3}\sqrt{\pi}$, etc. These special values serve to show that the graph of the function $y=\Gamma(x)$ has the general form given in Fig. 63, the number of branches for negative values of x being infinite.

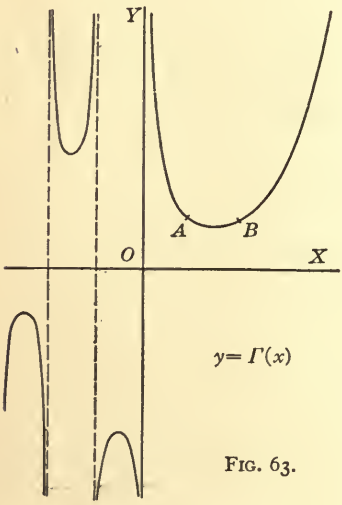

FIG. 63.

By equations (3) and (5), the Γ-function of any number is readily expressed in terms of a value of $\Gamma(n)$, in which n is between 1 and 2; it is therefore only necessary to tabulate $\Gamma(n)$ for this interval, corresponding to the arc AB in the diagram.

For purposes of computation, a table of values of $\log_{10}\Gamma(n)$ is more useful. Such a table, carried to 12 decimal places, was constructed by Legendre.* An abridgement of this table to 5 decimal places is given at the end of this chapter.

* *Traité des Fonctions Elliptiques et des Intégrales Eulériennes*, vol. ii, pp 490–499.

Transformations of the Eulerian Integrals.

348. A variety of definite integral forms of the Gamma function results from changes in the current variable, with corresponding changes in the limits. For example, putting $z=e^{-x}$, whence $x=-\log z$, we find

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx=\int_0^1(-\log z)^{n-1}dz=\int_0^1\left(\log\frac{1}{z}\right)^{n-1}dz.* \quad (1)$$

Again, putting $x=z^2$, we have

$$\Gamma(n)=2\int_0^\infty e^{-z^2}z^{2n-1}dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This form gives a direct evaluation of $\Gamma(\frac{1}{2})$; for, by Art. 288,

$$\Gamma(\tfrac{1}{2})=2\int_0^\infty e^{-z^2}dz=\sqrt{\pi},$$

agreeing with Art. 347.

Putting $x=\alpha y$, where α is any positive quantity, we find

$$\Gamma(n)=\alpha^n\int_0^\infty y^{n-1}e^{-\alpha y}dy. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

349. If, in the first Eulerian integral, we put $x=1/z$, we have

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_1^\infty\frac{(z-1)^{n-1}}{z^{m+n}}dz. \quad . \quad . \quad (1)$$

Again, putting $z=y+1$ in the last member,

$$B(m,n)=\int_0^\infty\frac{y^{n-1}}{(y+1)^{m+n}}dy.† \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* The last member is the original form in which Euler and Legendre treated the Gamma function.

† It will be found that substituting the reciprocal in this form merely interchanges m and n. The three forms in equations (1) and (2) are the simplest forms of $B(m,n)$. See also Ex. 24 below.

Putting x/a for the current variable in each case, these three forms are rendered homogeneous in x and a; the results are

$$\int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}B(m, n), \quad . \quad . \quad . \quad . \quad (3)$$

$$\int_a^\infty \frac{(x-a)^{n-1}}{x^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(x+a)^{m+n}}dx = \frac{B(m, n)}{a^m}. \quad . \quad . \quad (4)$$

A useful transformation results from putting $x=\sin^2\theta$, whence $1-x=\cos^2\theta$ and $dx=2\sin\theta\cos\theta\,d\theta$. We thus find

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta, \quad . \quad . \quad . \quad (5)$$

in which m and n may have any positive values.

350. In the special case where $m+n=1$, the B-function in the form (2) has already been integrated; thus, by equation (2), Art. 302,

$$B(m, 1-m) = \frac{\pi}{\sin m\pi}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $0<m<1$. Using the integral forms in equations (1) and (5) above, we have therefore also

$$\int_0^1 \frac{dx}{x^{1-m}(1-x)^m} = \int_1^\infty \frac{dz}{z(z-1)^m} = \frac{\pi}{\sin m\pi}, \quad . \quad . \quad . \quad (2)$$

and

$$\int_0^{\frac{\pi}{2}} \tan^{2m-1}\theta\,d\theta = \frac{\pi}{2\sin m\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As a particular case, putting $m=\frac{1}{4}$ or $\frac{3}{4}$ in equation (3),

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\tan\theta}} = \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta = \frac{\pi}{\sqrt{2}}.$$

Relation between the two Eulerian Integrals.

351. The product of two Γ-functions, when put in the double integral form, gives rise to a relation between the two Eulerian Integrals. We shall derive this relation by two methods analogous to those employed in the evaluation of k^2 in Arts. 289 and 288 respectively.

Taking α as the current variable, we have

$$\Gamma(m) = \int_0^\infty \alpha^{m-1} e^{-\alpha} d\alpha.$$

Multiplying each element by the constant $\Gamma(n)$ in the form, see Art. 348,

$$\Gamma(n) = \alpha^n \int_0^\infty x^{n-1} e^{-\alpha x} dx,$$

we have

$$\Gamma(m)\Gamma(n) = \int_0^\infty \alpha^{m+n-1} e^{-\alpha} \int_0^\infty x^{n-1} e^{-\alpha x} dx \, d\alpha.$$

Reversing the order of integration, we have

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty \alpha^{m+n-1} e^{-(x+1)\alpha} d\alpha \, x^{n-1} dx.$$

Performing the α-integration by equation (3), Art. 348,

$$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(x+1)^{m+n}} x^{n-1} dx = \Gamma(m+n) \int_0^\infty \frac{x^{n-1} dx}{(x+1)^{m+n}}.$$

Hence, by equation (2), Art. 349,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \quad \ldots\ldots\ (1)$$

352. In the other method of deriving this relation we take two independent variables x and y, and use the form (2), Art. 348. Thus

$$\Gamma(m)\Gamma(n)=4\int_0^\infty\int_0^\infty e^{-x^2-y^2}x^{2m-1}y^{2n-1}dx\,dy.$$

Then putting $x=r\cos\theta$ and $y=r\sin\theta$,

$$\Gamma(m)\Gamma(n)=4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2m+2n-1}e^{-r^2}\cos^{2m-1}\theta\sin^{2n-1}\theta\,dr\,d\theta$$

$$=2\int_0^{\frac{\pi}{2}}\Gamma(m+n)\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta.$$

Hence, by equation (5), Art. 349,

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

as before. When $m+n=1$, this equation gives, by means of equation (6), Art. 346, another proof of equation (1) of Art. 350.

Reduction of Integrals to Gamma Functions.

353. By means of the equation proved above, equation (5), Art. 349, may be written

$$\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)};$$

putting in this $2m-1=p$, $2n-1=q$, it becomes

$$\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},$$

in which, since m and n must be positive, p and q must each exceed -1. This is a generalization of formula (Q), p. 126, with which it can be shown to agree in the several cases where p and q are integers.

In its application, the arguments of the Γ-functions are to be reduced by the formula $\Gamma(r)=(r-1)\Gamma(r-1)$ or $\Gamma(r)=\frac{1}{r}\Gamma(r+1)$ until they fall within the limits of the table. For example, to compute the integral when $p=\frac{5}{3}$ and $q=\frac{2}{3}$, we have $\frac{1}{2}(p+1)=\frac{4}{3}$, $\frac{1}{2}(q+1)=\frac{5}{6}$. Then

$$V=\int_0^{\frac{\pi}{2}} \cos^{\frac{5}{3}}\theta \sin^{\frac{2}{3}}\theta\, d\theta=\frac{\Gamma(\frac{4}{3})\Gamma(\frac{5}{6})}{2\Gamma(\frac{13}{6})}=\frac{\frac{6}{5}\Gamma(\frac{4}{3})\Gamma(\frac{11}{6})}{2\cdot\frac{7}{6}\Gamma(\frac{7}{6})}$$
$$=\frac{18\Gamma(1.333\ldots)\Gamma(1.833\ldots)}{35\Gamma(1.166\ldots)}.$$

Using the table on p. 401, the computation takes the form

$$\begin{aligned}
\log 18&=1.25527\\
\log \Gamma(1.333\tfrac{1}{3})&=9.95084-10\\
\log \Gamma(1.833\tfrac{1}{3})&=9.97343-10\\
\operatorname{colog} 35&=8.45593-10\\
\operatorname{colog} \Gamma(1.166\tfrac{2}{3})&=0.03258\\
\hline
\log V&=9.66805-10 \qquad \therefore\ V=0.46564.
\end{aligned}$$

Besides the case in which p and q are both integers, there are two others in which the result is free from Γ-functions. These are, first when p or q is an odd integer, in which case one of the Γ's is a factorial and the others after reduction divide out. (Compare Ex. VII, 18.) Secondly when $p+q$ is an even integer, in which case advantage may be taken of equation (6), Art. 346. For example,

$$\int_0^{\frac{\pi}{2}} \cos^{\frac{1}{6}}\theta \sin^{\frac{2}{3}}\theta\, d\theta=\frac{\Gamma(\frac{7}{6})\Gamma(\frac{5}{6})}{2\Gamma(2)}=\frac{1}{12}\Gamma(\tfrac{1}{6})\Gamma(\tfrac{5}{6})=\frac{\pi}{12\sin\frac{1}{6}\pi}=\frac{\pi}{6}.$$

354. The integrals $\int_0^\infty \frac{\cos bz}{z^n}dz$ and $\int_0^\infty \frac{\sin bz}{z^n}dz$, where $0<n<1$, can be evaluated by forming, as in Art. 351, a double integral, and then reversing the order of integration. Thus, putting U for the first of these integrals, and multiplying its elements by $\Gamma(n)$ in the form

$$\Gamma(n)=z^n\int_0^\infty e^{-xz}x^{n-1}dx,$$

equation (3), Art. 348, we have

$$\Gamma(n)U=\int_0^\infty \cos bz\int_0^\infty e^{-xz}x^{n-1}dx\,dz.$$

Reversing the order, the z-integration can be performed by Art. 281, and then the x-integration by equation (4), Art. 302. Thus

$$\Gamma(n)U=\int_0^\infty x^{n-1}\int_0^\infty e^{-xz}\cos bz\,dz\,dx=\int_0^\infty \frac{x^n dx}{x^2+b^2},$$

and putting $x=by$,

$$U=\int_0^\infty \frac{\cos bz}{z^n}dz=\frac{b^{n-1}}{\Gamma(n)}\int_0^\infty \frac{y^n dy}{y^2+1}=\frac{\pi b^{n-1}}{2\Gamma(n)\cos\frac{1}{2}n\pi}. \quad . \quad (1)$$

In like manner, putting $V=\int_0^\infty \frac{\sin bz}{z^n}dz$ and supposing b positive, we find

$$\Gamma(n)V=\int_0^\infty x^{n-1}\int_0^\infty e^{-xz}\sin bz\,dz\,dx=\int_0^\infty \frac{bx^{n-1}dx}{x^2+b^2}$$

$$=b^{n-1}\int_0^\infty \frac{y^{n-1}dy}{y^2+1}=\frac{\pi b^{n-1}}{2\sin\frac{1}{2}n\pi},$$

or

$$\int_0^\infty \frac{\sin bz}{z^n}dz=\frac{\pi b^{n-1}}{2\Gamma(n)\sin\frac{1}{2}n\pi}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If b is negative, this integral changes sign. In the limiting case when $n=1$, the result agrees with Art. 283.

355. Two integrals of a more general form result from the substitution of a complex value for the constant α in equation (3), Art. 348. Thus, putting $\alpha=a-ib$ in

$$\Gamma(m)=\alpha^m\int_0^\infty e^{-\alpha x}x^{m-1}dx,$$

we have, since $e^{ib}=\cos bx+i\sin bx$,

$$\int_0^\infty e^{-ax}(\cos bx+i\sin bx)x^{m-1}dx=\frac{\Gamma(m)}{(a-ib)^m}=\frac{\Gamma(m)}{\rho^m}(\cos\theta+i\sin\theta)^m,$$

where $a+ib=\rho(\cos\theta+i\sin\theta)$, so that

$$\rho=\sqrt{(a^2+b^2)}\quad\text{and}\quad\theta=\tan^{-1}b/a.$$

Expanding by DeMoivre's Theorem, and separating real and imaginary parts,

$$\int_0^\infty e^{-ax}\cos bx\,.\,x^{m-1}dx=\frac{\Gamma(m)}{(a^2+b^2)^{\frac{m}{2}}}\cos\left[m\tan^{-1}\frac{b}{a}\right],\quad(1)$$

and

$$\int_0^\infty e^{-ax}\sin bx\,.\,x^{m-1}dx=\frac{\Gamma(m)}{(a^2+b^2)^{\frac{m}{2}}}\sin\left[m\tan^{-1}\frac{b}{a}\right].\quad(2)$$

In these equations, a is positive and $m>0$. They include the general equations given in Exs. XIX. 6 and 8.

356. When $a=0$, equations (1) and (2) become

$$\int_0^\infty x^{m-1}\cos bx\,dx=\frac{\Gamma(m)}{b^m}\cos m\frac{\pi}{2}$$

and

$$\int_0^\infty x^{m-1} \sin bx\, dx = \frac{\Gamma(m)}{b^m} \sin m\frac{\pi}{2},$$

but in this case we must have $m<1$.

Putting $m+n=1$, these may be written

$$\int_0^\infty \frac{\cos bx}{x^n}\, dx = \Gamma(1-n) b^{n-1} \sin n\frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\sin bx}{x^n}\, dx = \Gamma(1-n) b^{n-1} \cos n\frac{\pi}{2},$$

in which, since m cannot be negative, n cannot exceed unity. Since $\Gamma(1-n)=\dfrac{\pi}{\Gamma(n)\sin n\pi}$, these last equations are identical with those of Art. 354.

Reduction of Certain Multiple Integrals.

357. The double integral of $x^{l-1}y^{m-1}dy\,dx$, where x and y have all positive values such that $x+y<a$ is expressible in Γ-functions. Thus

$$\int_0^a \int_0^{a-x} x^{l-1}y^{m-1}dy\,dx = \frac{1}{m}\int_0^a x^{l-1}(a-x)^m dx = \frac{a^{l+m}B(l,\, m+1)}{m},$$

by equation (3), Art. 349. Hence, by equation (1), Art. 351,

$$\int_0^1 \int_0^{1-x} x^{l-1}y^{m-1}dy\,dx = \frac{a^{l+m}\Gamma(l)\Gamma(m+1)}{m\Gamma(l+m+1)} = \frac{a^{l+m}\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)} \quad . \quad . \quad (1)$$

This result may be extended to any number of variables. Thus, for three variables, putting

$$U=\int_0^a\int_0^{a-x}\int_0^{a-x-y}x^{l-1}y^{m-1}z^{n-1}dz\,dy\,dx,$$

the y- and z-integrations can be performed by equation (1), $a-x$ taking the place of a; hence

$$U=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\int_0^a x^{l-1}(a-x)^{m+n}dx$$

$$=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\,\frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)}a^{l+m+n}$$

by equation (3), Art. 349. Therefore, if x, y and z are three variables subject to the condition that each is positive and their sum shall not exceed a,

$$\iiint x^{l-1}y^{m-1}z^{n-1}dz\,dy\,dx=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}a^{l+m+n}. \quad . \quad (2)$$

It is obvious that this result can, in like manner, be extended to any number of variables subject to the same conditions. Thus

$$U=\iiint\ldots x^{l-1}y^{m-1}z^{n-1}\ldots dz\,dy\,dx \quad - - -$$

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)\cdots}{\Gamma(l+m+n+\cdots+1)}a^{l+m+n+\cdots}. \quad . \quad . \quad . \quad (3)$$

358. By means of equation (3), an integral of the form

$$V=\iiint\cdots x^{l-1}y^{m-1}z^{n-1}\cdots f(x+y+z+\cdots)\cdots dz\,dy\,dx$$

can at once be reduced to a simple integral.

For this purpose, we establish the following general theorem: Let δU denote the element of a multiple integral U, and let V be the integral under the same limits of the element $f(\alpha)\delta U$, where α is a function of the variables of integration: then *if U is expressible as a function of α*, we shall have, with proper limits for α,

$$V = \int U' f(\alpha) d\alpha, \quad \text{where} \quad U' = \frac{dU}{d\alpha}.$$

To prove this, imagine the multiple integrals U and V to be transformed to any new set of variables of which α is one, and suppose all the integrations except that for α to have been performed. The expression under the final integral sign will, by hypothesis, be $U'd\alpha$ in the first case. Now, since in these integrations α is treated as a constant, the integrations in the second case will not be affected by the presence of the factor $f(\alpha)$; hence the expression under the final integral sign will be $U' f(\alpha) d\alpha$; therefore $V = \int U' f(\alpha) d\alpha$, as stated above.*

In the present case, U is, by equation (3), a function of a, and putting for abridgment

$$N = \frac{\Gamma(l)\Gamma(m)\cdots}{\Gamma(l+m+\cdots)}, \qquad \frac{dU}{da} = (l+m+\cdots) N a^{l+m+\cdots+1};$$

hence, by the theorem

$$V = (l+m+\cdots) N \int_0^a f(a) a^{l+m+\cdots+1} da.$$

* In further explanation of the principle employed above, see Art. 222, where it is applied to a problem of mean values. In that article N, "the number of cases," corresponds to U, and MN "the aggregate" to V; N is a known function of r (which corresponds to α), and M_0 (also a function of r) corresponds to $f(\alpha)$. Then, when the independent variable r is increased by dr, the number of "new cases" is dN and the new part of the aggregate is $M_0 dN$; whence MN is obtained by integration of $M_0 dN$ with respect to r.

The Function Log $\Gamma(1+x)$, and Euler's Constant.

359. By equation (1), Art. 344,

$$\Pi(x)=\Gamma(1+x)=\left[n^x \frac{1}{1+x}\frac{1}{1+\frac{1}{2}x}\cdots\frac{1}{1+\frac{1}{n}x}\right]_{n=\infty};$$

whence

$$\log\Gamma(1+x)=\left[x\log n-\log(1+x)-\log(1+\tfrac{1}{2}x)-\cdots-\log\left(1+\frac{1}{n}\right)x\right]_{n=\infty}. \quad (1)$$

Developing the logarithms by the series

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\cdots,$$

and collecting like terms, we find

$$\log\Gamma(1+x)=x\left[\log n-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right]_{n=\infty}$$
$$+\frac{x^2}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)-\frac{x^3}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right)+\cdots$$
$$+(-1)^r\frac{x^r}{r}\left(1+\frac{1}{2^r}+\frac{1}{3^r}+\cdots\right)+\cdots$$

The logarithmic series employed are convergent when $x<1$, and the numerical series are all convergent with the exception of that which occurs in the coefficient of x, which is known as *the harmonic series* and is shown in Diff. Calc., Art. 180, to have an infinite value. Thus the coefficient of x, which takes the form

$\infty - \infty$, must have a finite value. The numerical value of this coefficient (which is found to be negative) is known as *Euler's Constant* and is generally denoted by γ. Thus

$$\gamma = \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right]_{n=\infty};$$

its value will be found in Art. 362.

360. Putting, when $n > 1$,

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots,*$$

the development of $\log \Gamma(1+x)$ may now be written

$$\log \Gamma(1+x) = -\gamma x + \tfrac{1}{2} S_2 x^2 - \tfrac{1}{3} S_3 x^3 + \cdots \quad . \quad . \quad (1)$$

Again, since the sign of x may be changed in the expression for $\Pi(x)$, Art. 344, we have in like manner

$$\log \Gamma(1-x) = \gamma x + \tfrac{1}{2} S_2 x^2 + \tfrac{1}{3} S_3 x^3 + \cdots \quad . \quad . \quad . \quad (2)$$

The sum of the terms containing even powers of x in these series can be at once evaluated. For, by equations (3) and (6), Art. 346,

$$\Gamma(1+x)\Gamma(1-x) = x\Gamma(x)\Gamma(1-x) = \frac{\pi x}{\sin \pi x};$$

therefore the sum of equations (1) and (2) gives

$$\log \frac{\pi x}{\sin \pi x} = S_2 x^2 + \tfrac{1}{2} S_4 x^4 + \tfrac{1}{3} S_6 x^6 + \cdots \dagger. \quad . \quad . \quad (3)$$

* Compare Diff. Calc., Art. 239. In Art. 244 the even-numbered S's are shown to be connected with Bernoulli's numbers.

† This equation is identical with equation (2), Diff. Calc., Art. 239, of which

Substituting in equation (1),

$$\log \Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x}{\sin \pi x}-(\gamma x+\tfrac{1}{3}S_3x^3+\tfrac{1}{5}S_5x^5+\cdots), \qquad (4)$$

in which the sign of x may be changed, and the limits of convergence are ± 1.

361. The values of S_n all exceed the first term, which is unity; hence the series (4) converges slowly. But, if we put $S_n=1+s_n$, the part corresponding to the first term takes a known form. Thus, substituting in equation (4), we have

$$\log \Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x}{\sin \pi x}-\gamma x-[\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\cdots]$$
$$-[\tfrac{1}{3}s_3x^3+\tfrac{1}{5}s_5x^5+\cdots],$$

in which the sum of the first series is, by Diff. Calc., Art. 198, $\frac{1}{2}\log\frac{1+x}{1-x}-x$. Hence

$$\log \Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x(1-x)}{(1+x)\sin \pi x}+(1-\gamma)x$$
$$-[\tfrac{1}{3}s_3x^3+\tfrac{1}{5}s_5x^5+\cdots] \qquad (5)$$

and

$$\log \Gamma(1-x)=\tfrac{1}{2}\log\frac{\pi x(1+x)}{(1-x)\sin \pi x}-(1-\gamma)x$$
$$+[\tfrac{1}{3}s_3x^3+\tfrac{1}{5}s_5x^5+\cdots].* \qquad (6)$$

the differential [equation (2), Art. 243] gives the values of S_{2n} in terms of Bernoulli's numbers.

* By means of these series Legendre (having first calculated the values of S_n to 16 decimal places, together with the value of γ) calculated the values of $\log \Gamma(a)$. Since it was only necessary to cover a range of unity in the values of the argument, only values of x less than $\frac{1}{2}$ had to be used, and in fact other relations were used to limit the range still further. See, for example, Ex. 4 below.

362. When known values of the Γ-function are substituted in these equations, relations between γ and the values of s_n are found. Thus, putting $x=1$ in equation (5), we have, by evaluation of an indeterminate form,

$$\gamma=1-\tfrac{1}{2}\log 2-[\tfrac{1}{3}s_3+\tfrac{1}{5}s_5+\tfrac{1}{7}s_7+\cdots] \quad . \quad . \quad . \quad . \quad (7)$$

Again, by putting $x=\frac{1}{2}$ in equation (6), we derive, since $\Gamma(\frac{1}{2})=\sqrt{\pi}$,

$$\gamma=1-\log\frac{3}{2}-\left[\frac{s_3}{3.2^2}+\frac{s_5}{5.2^4}+\frac{s_7}{7.2^6}+\cdots\right]. \quad . \quad . \quad (8)$$

A table of the values of s_n is given at the end of this chapter, by means of which the value of γ * to 10 places of decimals will be found to be

$$\gamma=0.5772156649.$$

The Logarithmic Derivative of $\Gamma(x)$.

363. Let $\Gamma'(x)$ denote the derivative of $\Gamma(x)$, and $\psi(x)$ the logarithmic derivative; then, differentiating equation (1), Art. 359 (and putting x in place of $1+x$), we have

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\left[\log n-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\cdots-\frac{1}{x+n-1}\right]_{n=\infty}. \quad (1)$$

* The value of γ has been calculated by Professor J. C. Adams to 263 places of decimals, *Proceedings of the Royal Society of London*, vol. xxvii, p. 94. The method involves the direct summation of a large number of terms of the harmonic series and calculation of the remainder by means of Euler's formula for the summation of series. The same method was used by Euler and Legendre for S_2, S_3, etc.; it is in fact the only practicable method for small values of n, owing to the slow convergence of the series. See Boole's Finite Differences, 2d Edition, p. 93.

It follows that if r is a positive integer,

$$\psi(x+r)=\psi(x)+\frac{1}{x}+\frac{1}{x+1}+\cdots+\frac{1}{x+r-1}. \quad . \quad . \quad (2)$$

This equation may also be derived by successive steps from the logarithmic differentiation of $\Gamma(x+1)=x\Gamma(x)$. It shows that the sum of a finite number of terms of any harmonic series can be expressed in terms of ψ-functions, that is in terms of Γ- and Γ'-functions.

Putting $x=1$ in equation (1), or $x=0$ in the derivative of equation (1), Art. 360, we find $\Gamma'(1)=-\gamma$. Hence from equation (2) we have, when r is a positive integer,

$$\psi(r+1)=-\gamma+1+\frac{1}{2}+\cdots+\frac{1}{r};$$

and, since $\Gamma(r+1)=r!$,

$$\Gamma'(r+1)=r!\left(-\gamma+1+\frac{1}{2}+\cdots+\frac{1}{r}\right).$$

In particular $\Gamma'(1)=-\gamma$, and $\Gamma'(2)=1-\gamma$; these give the slope of the graph of $\Gamma(x)$, p. 378, at the points A and B. Again, when r is very great, we have approximately, from equation (1) and the definition of γ, $\Gamma'(r)/\Gamma(r)=\log r$, which shows that the subtangent of this curve tends to a ratio of equality to the Napierian logarithm of the abscissa.

364. An important theorem due to Gauss follows readily from equation (1); for, putting mx in place of x, where m is an integer, it becomes

$$\frac{d\log\Gamma(mx)}{m\,dx}=\left[\log n-\frac{1}{mx}-\frac{1}{mx+1}-\cdots-\frac{1}{mx+n-1}\right]_{n=\infty}.$$

Multiplying by m, the terms may be written in groups thus:

$$\frac{d\log\Gamma(mx)}{dx}=m\log n-\frac{1}{x}-\frac{1}{x+\frac{1}{m}}-\frac{1}{x+\frac{2}{m}}-\cdots-\frac{1}{x+\frac{m-1}{m}}$$

$$-\frac{1}{x+1}-\frac{1}{x+1+\frac{1}{m}}-\frac{1}{x+1+\frac{2}{m}}-\cdots-\frac{1}{x+1+\frac{m-1}{m}}$$

$$-\frac{1}{x+2}-\frac{1}{x+2+\frac{1}{m}}-\cdots$$

$$\vdots\qquad\vdots$$

The number of columns is m and the number of terms in each is n', where $n=mn'$ and n' is to be made infinite. Now, since $m\log n=m\log n'+m\log m$, we can assign $\log n'$ to each of the m columns, and then sum the column by equation (1). Thus we have

$$\frac{d}{dx}\log\Gamma(mx)=m\log m+\frac{d}{dx}\log\Gamma(x)+\frac{d}{dx}\log\Gamma\left(x+\frac{1}{m}\right)$$
$$+\frac{d}{dx}\log\Gamma\left(x+\frac{2}{m}\right)+\cdots+\frac{d}{dx}\log\Gamma\left(x+\frac{m-1}{m}\right).$$

Integrating this equation,

$$\log\Gamma(mx)=C+mx\log m+\log\Gamma(x)+\log\Gamma\left(x+\frac{1}{m}\right)+\cdots$$
$$+\log\Gamma\left(x+\frac{m-1}{m}\right),$$

and taking the exponential of each side,

$$\Gamma(mx) = Am^{mx}\Gamma(x)\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\ldots\Gamma\left(x+\frac{m-1}{m}\right), \quad (3)$$

where A is a constant of integration to be determined.

365. For this purpose, we notice that, putting $x = 1/m$,

$$1 = mA\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{3}{m}\right)\ldots\Gamma\left(\frac{m-1}{m}\right)\ldots \quad (4)$$

Now, by equation (6), Art. 346,

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{m-1}{m}\right) = \frac{\pi}{\sin\frac{\pi}{m}},$$

and, taking in like manner the products of pairs of factors equidistant from the ends,

$$\left[\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\ldots\Gamma\left(\frac{m-1}{m}\right)\right]^2 = \frac{\pi^{m-1}}{\sin\frac{\pi}{m}\sin 2\frac{\pi}{m}\cdots\sin(m-1)\frac{\pi}{m}}.$$

This, by equation (3), Diff. Calc., Art. 234, gives

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{3}{m}\right)\ldots\Gamma\left(\frac{m-1}{m}\right) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \quad (5)$$

Substituting in equation (4), we find

$$\frac{1}{A} = \sqrt{m}(2\pi)^{\frac{m-1}{2}},$$

and this in equation (3) gives Gauss' theorem,

$$\Gamma(x)\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\ldots\Gamma\left(x+\frac{m-1}{m}\right)$$
$$=\Gamma(mx)(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mx}. \quad . \quad (6)$$

Equation (5) is a special case of this equation.

Examples XXII.

1. Show that

$$\int_0^1 \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\frac{1}{n}B\left(\frac{p}{n},\frac{q}{n}\right).$$

The first member is the original form in which Euler introduced the integral, p, q, and n being integers. Legendre thus reduced all cases to that in which $n=1$, by introducing fractional values of the arguments.

2. Prove that

$$\int_0^1 \frac{x^{n-1}+x^{m-1}}{(1+x)^{m+n}}dx=\int_8^1 \frac{x^{n-1}+x^{m-1}}{(1+x)^{m+n}}dx=B(m,n).$$

3. Prove that

$$B(m,m)=\frac{B(m,\tfrac{1}{2})}{2^{2m-1}}.$$

Put $x=\frac{1}{2}(1+y)$, and then $y^2=z$.

4. From example 3 derive the general property of the Γ-function

$$\Gamma(\tfrac{1}{2}+m)\Gamma(m)=2^{1-2m}\sqrt{\pi}\Gamma(2m).$$

Supposing the values of $\Gamma(x)$ from $x=0$ to $x=\frac{1}{3}$ to have been found, those from $\frac{1}{2}$ to $\frac{2}{3}$ can be derived from this formula, and thence by means of equation (6), *Art.* 346, *all values of $\Gamma(x)$.*

5. Derive the continued product

$$\frac{\{\Gamma(n)^2\}}{\Gamma(n+m)\Gamma(n-m)} = \left[1-\frac{m^2}{n^2}\right]\left[1-\frac{m^2}{(n+1)^2}\right]\left[1-\frac{m^2}{(n+2)^2}\right]\cdots$$

6. Show that

$$\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} = \frac{\Gamma(n)}{m^n}.$$

7. Prove that

$$B(m, n)B(m+n, l) = B(m, l)B(m+l, n).$$

8. Derive the fundamental property of the Γ-function by differentiation of $\int_0^\infty x^{n-1}e^{-ax}dx$ with respect to a, and also by integration with respect to a.

Reduce the following integrals to Γ-functions:

9. $\int_0^1 \frac{dz}{\sqrt{(1-z^4)}}$. $\quad \frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{4\Gamma(\frac{3}{4})} = 1.3110.$

10. $\int_1^\infty (\log z)^n \frac{dz}{z^m}$. $\quad \frac{\Gamma(n+1)}{(m-1)^{n+1}}.$

11. $\int_0^\pi \frac{\cos^2 x}{\sqrt{(\sin x)}}dx.$ $\quad 3.496.$

12. $\int_0^\infty \frac{x^{p-1}dx}{(ax+b)^{p+q}}.$ $\quad \frac{\Gamma(p)\Gamma(q)}{a^p b^q \Gamma(p+q)}.$

13. $\int_0^{\frac{\pi}{2}} \frac{\sin^{2p-1}\theta\cos^{2q-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{p+q}}.$ $\quad \frac{\Gamma(p)\Gamma(q)}{2a^p b^q \Gamma(p+q)}.$

14. $\int_0^\infty e^{-a^m x^m}dx.$ $\quad \frac{1}{ma}\Gamma\left(\frac{1}{m}\right).$

15. $\int_0^\infty \frac{e^{-\frac{a}{x}}dx}{x^n}, \quad n>1.$ $\quad \frac{\Gamma(n-1)}{a^{n-1}}.$

16. $\int_0^1 x^{l-1}(1-x^2)^{m-1}dx.$ $\dfrac{\Gamma(\frac{1}{2}l)\Gamma(m)}{2\Gamma(\frac{1}{2}l+m)}.$

17. $\int_0^\infty \cos(bx^n)dx, \quad (n>0).$ $\dfrac{\Gamma\left(1+\frac{1}{n}\right)}{b^{\frac{1}{n}}}\cos\dfrac{\pi}{2n}.$

18. Deduce equation (3), Art. 350, from (6), Art. 346, by using the form of Γ given in equation (2), Art. 348.

19. Show that, when $n=0$, $(\beta^n-\alpha^n)\Gamma(n)=\log\dfrac{\beta}{\alpha}$.

20. Find the area in the first quadrant enclosed by the curve

$$\frac{x^n}{a^n}+\frac{y^n}{b^n}=1, \quad (n>0),$$

and apply to the cases $n=4$, $n=\infty$, and $n=\frac{1}{2}$.

$$\text{Area}=\frac{ab}{2n}\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}^2}{\Gamma\left(\frac{2}{n}\right)}=ab\frac{\left\{\Gamma\left(1+\frac{1}{n}\right)\right\}^2}{\Gamma\left(1+\frac{2}{n}\right)}.$$

$n=4$ gives $A=.927ab$; $n=\infty$, $A=ab$; $n=\frac{1}{2}$, $A=\frac{1}{6}ab$.

21. Find the volume generated by a loop of the lemniscate

$$r^2=a^2\sin 2\theta$$

revolving about the axis of x. $1.2337a^3$.

22. Find the length of a loop of the curve

$$r^{\frac{4}{5}}=a^{\frac{4}{5}}\cos\tfrac{4}{5}\theta. \qquad 3.3749a.$$

23. Express $\dfrac{1}{\sqrt{x}}$ as a definite z-integral by means of equation (2), Art. 289, and thence evaluate $\int_0^\infty \dfrac{\cos x}{\sqrt{x}}dx$ as a double integral. Compare Art. 354. $\sqrt{\dfrac{\pi}{2}}$.

24. Show that $B(m, n)$ may be transformed into

$$\frac{(a+b)^m b^n}{a^{m+n-1}} \int_0^a \frac{y^{m-1}(a-y)^{n-1}}{(y+b)^{m+n}} dy,$$

where a and b are positive; whence

$$\int_0^a \frac{y^{m-1}(a-y)^{n-1}}{(y+b)^{m+n}} dy = \frac{a^{m+n-1}\Gamma(m)\Gamma(n)}{(a+b)^m b^n \Gamma(m+n)}.$$

25. Find the value of the multiple integral

$$\iiint \ldots x^{l-1} y^{m-1} z^{n-1} \ldots dz\, dy\, dx$$

for all positive values of the variables, such that

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r + \ldots < 1.$$

$$\frac{a^l b^m c^n \ldots}{pqr\ldots} \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)\ldots}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+\ldots+1\right)}.$$

26. Show that

$$\psi(1+x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} = -\gamma + S_2 x - S_3 x^2 + S_4 x^3 - \ldots$$

$$= -\gamma + \frac{x}{1+x} + s_2 x - s_3 x^2 + s_4 x^3 - \ldots$$

$$= -\gamma + \frac{x}{1+x} + \frac{x}{2(2+x)} + s_2' x - s_3' x^2 + \ldots$$

$$= -\gamma + \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + s_2'' x - \ldots$$

$$= -\gamma + \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \ldots$$

where $s_n' = (\frac{1}{3})^n + (\frac{1}{4})^n + \ldots$, etc.

27. Prove that

$$\Gamma'(\tfrac{1}{2}) = -\sqrt{\pi}[1+\gamma+\tfrac{1}{2}s_2+\tfrac{1}{4}s_3+\tfrac{1}{8}s_4+\ldots] = -3.48023$$

and

$$\Gamma'(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}[\tfrac{1}{3}-\gamma+\tfrac{1}{2}s_2-\tfrac{1}{4}s_3+\tfrac{1}{8}s_4-\cdots] = 0.032338.$$

28. Prove by direct summation of the powers of the reciprocals that

$$s_2+s_3+s_4+s_5+\cdots = 1, \qquad \text{and} \qquad s_2-s_3+s_4-s_5+\cdots = \tfrac{1}{2}.$$

29. Derive the numerical series

$$\gamma = \tfrac{1}{2}S_2-\tfrac{1}{3}S_3+\tfrac{1}{4}S_4-\cdots;$$

and thence

$$\tfrac{1}{2}s_2-\tfrac{1}{3}s_3+\tfrac{1}{4}s_4-\cdots = \log 2+\gamma-1.$$

30. Derive the series

$$\log\frac{\pi x(1-x^2)}{\sin \pi x} = s_2x^2+\tfrac{1}{2}s_4x^4+\tfrac{1}{3}s_6x^6+\cdots;$$

and thence

$$\log 2 = s_2+\tfrac{1}{2}s_4+\tfrac{1}{3}s_6+\cdots$$

Hence also, from Ex. 29, we may derive

$$\gamma = 1-\tfrac{1}{2}s_2-\tfrac{1}{3}s_3-\tfrac{1}{4}s_4-\cdots$$

31. Express $\Gamma'(x)$ as a definite integral, giving particular results when $x=1$, $x=2$ and $x=3$.

$$\int_0^\infty e^{-z}\log z\,dz = -\gamma; \quad \int_0^\infty ze^{-z}\log z\,dz = 1-\gamma; \quad \int_0^\infty z^2e^{-z}\log z\,dz = 3-2\gamma.$$

LOG $\Gamma(n)$.

n	0	1	2	3	4	5	6	7	8	9
1.00	0.	9.99975	950	925	900	876	851	826	802	777
1.01	9.99753	729	704	680	656	632	608	584	560	536
1.02	513	489	466	442	419	395	372	349	326	303
1.03	280	257	234	211	188	166	143	121	098	076
1.04	053	031	009	*987	*965	*943	*921	*899	*877	*855
1.05	9.98834	812	791	769	748	727	705	684	663	642
1.06	621	600	579	558	538	517	496	476	455	435
1.07	415	394	374	354	334	314	294	274	254	234
1.08	215	195	175	156	137	117	098	079	059	040
1.09	021	002	*983	*964	*946	*927	*908	*890	*871	*853
1.10	9.97834	816	797	779	761	743	725	707	689	671
1.11	653	635	618	600	582	565	548	530	513	496
1.12	478	461	444	427	410	393	376	360	343	326
1.13	310	293	277	260	244	228	211	195	179	163
1.14	147	131	115	099	083	068	052	036	021	005
1.15	9.96990	975	959	944	929	914	899	884	869	854
1.16	839	824	810	795	780	766	751	737	722	708
1.17	694	680	666	651	637	623	610	596	582	568
1.18	554	541	527	514	500	487	473	460	447	434
1.19	421	407	394	381	369	356	343	330	317	305
1.20	292	280	267	255	242	230	218	206	194	181
1.21	169	157	146	134	122	110	098	087	075	064
1.22	052	041	029	018	007	*995	*984	*973	*962	*951
1.23	9.95940	929	918	908	897	886	876	865	854	844
1.24	833	823	813	803	792	782	772	762	752	742
1.25	732	722	712	703	693	683	674	664	655	645
1.26	636	627	617	608	599	590	581	572	563	554
1.27	545	536	527	519	510	501	493	484	476	467
1.28	459	451	442	434	426	418	410	402	394	386
1.29	378	370	362	355	347	339	332	324	317	309
1.30	302	295	287	280	273	266	259	252	245	238
1.31	231	224	217	211	204	197	191	184	178	171
1.32	165	159	152	146	140	134	127	121	115	109
1.33	104	098	092	086	080	075	069	063	058	052
1.34	047	042	036	031	026	020	015	010	005	000
1.35	9.94995	990	985	981	976	971	966	962	957	953
1.36	948	944	939	935	930	926	922	918	914	910
1.37	905	901	898	894	890	886	882	878	875	871
1.38	868	864	861	857	854	850	847	844	840	837
1.39	834	831	828	825	822	819	816	813	811	808

LOG $\Gamma(n)$—(*Continued*).

n	0	1	2	3	4	5	6	7	8	9
1·40	9·94805	803	800	797	795	793	790	788	785	783
1·41	781	779	776	774	772	770	768	766	764	763
1·42	761	759	757	756	754	752	751	749	748	747
1·43	745	744	743	741	740	739	738	737	736	735
1·44	734	733	732	731	730	730	729	728	728	727
1·45	727	726	726	725	725	725	725	724	724	724
1·46	724	724	724	724	724	724	724	725	725	725
1·47	725	726	726	727	727	728	728	729	730	730
1·48	731	732	733	733	734	735	736	737	738	740
1·49	741	742	743	744	746	747	748	750	751	753
1·50	754	756	758	759	761	763	765	767	768	770
1·51	772	774	776	779	781	783	785	787	790	792
1·52	794	797	799	802	804	807	809	812	815	817
1·53	820	823	826	829	832	835	838	841	844	847
1·54	850	853	856	860	863	866	870	873	877	880
1·55	884	887	891	895	898	902	906	910	914	917
1·56	921	925	929	933	938	942	946	950	954	959
1·57	963	967	972	976	981	985	990	994	999	*004
1·58	9·95008	013	018	023	027	032	037	042	047	052
1·59	057	062	068	073	078	083	089	094	099	105
1·60	110	116	121	127	132	138	144	149	155	161
1·61	167	173	179	185	190	196	203	209	215	221
1·62	227	233	240	246	252	259	265	271	278	285
1·63	291	298	304	311	318	324	331	338	345	352
1·64	359	366	373	380	387	394	401	408	415	423
1·65	430	437	445	452	459	467	474	482	489	497
1·66	505	512	520	528	536	543	551	559	567	575
1·67	583	591	599	607	615	624	632	640	648	657
1·68	665	673	682	690	699	707	716	724	733	742
1·69	750	759	768	777	785	794	803	812	821	830
1·70	839	848	857	866	876	885	894	903	913	922
1·71	931	941	950	960	969	979	988	998	*008	*017
1·72	9·96027	037	047	056	066	076	086	096	106	116
1·73	126	136	146	157	167	177	187	198	208	218
1·74	229	239	250	260	271	281	292	302	313	324
1·75	335	345	356	367	378	389	400	411	422	433
1·76	444	455	466	477	488	499	511	522	533	545
1·77	556	567	579	590	602	614	625	637	648	660
1·78	672	684	695	707	719	731	743	755	767	779
1·79	791	803	815	827	839	851	864	876	888	901

LOG $\Gamma(n)$—(*Continued*).

n	0	1	2	3	4	5	6	7	8	9
1.80	9.96913	925	938	950	963	975	988	*000	*013	*026
1.81	9.97038	051	064	076	089	102	115	128	141	154
1.82	167	180	193	206	219	232	245	259	272	285
1.83	298	312	325	339	352	366	379	393	406	420
1.84	433	447	461	474	488	502	516	530	543	557
1.85	571	585	599	613	627	641	656	670	684	698
1.86	712	727	741	755	770	784	798	813	827	842
1.87	856	871	886	900	915	930	944	959	974	989
1.88	9.98004	018	033	048	063	078	093	108	123	139
1.89	154	169	184	199	215	230	245	261	276	291
1.90	307	322	338	353	369	385	400	416	432	447
1.91	463	479	495	511	526	542	558	574	590	606
1.92	622	638	654	671	687	703	719	735	752	768
1.93	784	801	817	834	850	867	883	900	916	933
1.94	949	966	983	999	*016	*033	*050	*067	*083	*100
1.95	9.99117	134	151	168	185	202	219	237	254	271
1.96	288	305	323	340	357	375	392	409	427	444
1.97	462	479	497	515	532	550	567	585	603	621
1.98	638	656	674	692	710	728	746	764	782	800
1.99	818	836	854	872	890	909	927	945	963	982

VALUES OF $s_n = (\frac{1}{2})^n + (\frac{1}{3})^n + (\frac{1}{4})^n + \cdots$

n	s_n	n	s_n
2	0.64493 40668 5	14	0.00006 12481 4
3	.20205 69031 6	15	.00003 05882 4
4	.08232 32337 1	16	.00001 52822 6
5	.03692 77551 4	17	.00000 76372 0
6	.01734 30619 8	18	.00000 38172 9
7	.00834 92773 8	19	.00000 19082 1
8	.00407 73562 0	20	.00000 09539 6
9	.00200 83928 3	21	.00000 04769 3
10	.00099 45751 3	22	.00000 02384 5
11	.00049 41886 0	23	.00000 01192 2
12	.00024 60865 5	24	.00000 00596 1
13	.00012 27133 5	25	.00000 00298 0

N.B. For greater values of n, divide s_{25} successively by 2.

CHAPTER VI.

INTEGRATION OF FUNCTIONS OF THE COMPLEX VARIABLE.

XXIII.

Complex Values of the Derivative.

366. In treating of the functions of a complex variable it is usual to put

$$z=x+iy$$

for the independent variable; then if $w=f(z)$ it will usually be of the form $w=u+iv$, in which u and v are real quantities involving the real quantities x and y. It is explained in Diff. Calc., Art. 223, how a value of z is geometrically represented, in a plane of reference, by the point whose rectangular coordinates are x and y, and also that z may be put in the form $re^{i\theta}$, where r and θ are the polar coordinates of the same point, which we shall call the point z. In this last form r is called the modulus of z and is regarded as its *absolute value*, while θ is the argument or angle determining the direction of the unit factor $e^{i\theta}$, which is one of the radii of the unit circle.

In like manner, we may write

$$w=f(z)=u+iv=\rho e^{i\psi}$$

and represent w by a point referred to rectangular axes of u and v, generally taken for convenience in another plane which we may call the w-plane.

Thus the functional relation $w=f(z)$ establishes a correspondence between positions, P_1, P_2, P_3 etc., of the point (x, y) in the z-plane and positions, Q_1, Q_2, Q_3 etc., of the point (u, v) in the w-plane. Thus the w-plane becomes, as it were, a *map* of the z-plane, in which any figure described in the z-plane is represented by a figure which is called its *image*. The mode in which this mapping takes place constitutes the geometric representation of the function which gives rise to it, just as in the simpler case of the function of a real variable the curve $y=f(x)$ represents the function.

367. A continuous variation of the independent variable z is now represented by a continuous motion of the point z, which may be along any arbitrary path or track. The point w will then describe a corresponding track in the w-plane, which is the image of the track of z. Let z, represented by P in Fig. 64,

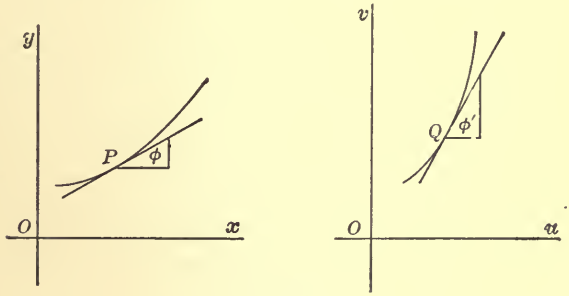

FIG. 64.

describe its track in a definite manner. Its motion at any instant involves a definite rate and a definite direction. Accordingly its differential,

$$dz=dx+i\,dy,$$

depends upon two independent arbitrary elements, the values of dx and dy, which measure in fact the resolved velocities of P

in the directions of the two axes. The actual velocity, or absolute rate of z, is measured by ds, the differential of the arc described. Denoting, as usual, by ϕ the inclination of the curve to the axis of x, we have

$$dz = dx + i\,dy = ds e^{i\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the last member, the rate and the direction of motion are expressed separately by the values of the independent quantities ds and ϕ; $ds = \sqrt{(dx^2 + dy^2)}$ being the modulus of dz, and ϕ its argument.

368. In like manner, we have, for the differential of the function,

$$dw = du + i\,dv = ds' e^{i\phi'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ds' in Fig. 64 being the absolute length or modulus of dw, and ϕ' its inclination to the axis of u. From equations (1) and (2) we obtain the derivative of the function $w = f(z)$, namely

$$f'(z) = \frac{dw}{dz} = \frac{du + i\,dv}{dx + i\,dy} = \frac{ds'}{ds} e^{i(\phi' - \phi)}. \quad . \quad . \quad . \quad . \quad (3)$$

The final member of this equation shows that the derivative is a complex quantity, of which *the modulus is the ratio of the rates, and the argument is the difference of the directions* of the motions of w and z. Since the derivative has in general a definite value for a given value of z, both this ratio of rates and this difference of direction are independent of the arbitrary path of z. Thus, if two paths of z start from the same point P, Fig. 65, making a given angle, $\phi_1 - \phi_2 = \alpha$, their images in the w-plane will make the same angle at Q, that is, $\phi_1' - \phi_2' = \alpha$.

369. It is an obvious consequence of this *preservation of angles* in the image that the image of a small area will be a similar small area (the ratio of similitude being the ratio of rates, which, as mentioned above, is the modulus of the derivative).

For this reason, the image is said to constitute a *conformal representation.**

Assuming the derivative to be finite and continuous, the scale of representation, or *magnification*, and the orientation change continuously from point to point. The points for which the

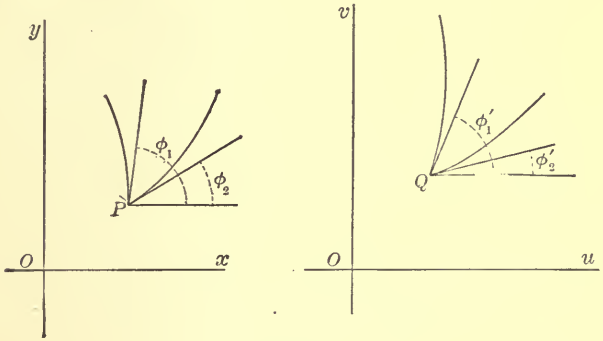

FIG. 65.

modulus of the derivative is zero or infinite (and its argument therefore indeterminate) are points of discontinuity for the function, and at these points the conformal representation fails.

Conjugate Functions of x and y.

370. When the function $w=f(z)=f(x+iy)$ is given in the form $u+iv$, u and v become known functions of the two real variables x and y. For example, if $f(z)=z^2$,

$$u+iv=x^2-y^2+2ixy;$$

whence

$$u=x^2-y^2, \qquad v=2xy.$$

* In like manner, a map of a spherical surface in which angles are preserved is a conformal representation of the surface. The stereographic projection is an example. In that case, the magnification has at every point of the circumference of the primitive circle a value double that at the centre.

In the general equation expressing the derivative, Art. 368,

$$f'(z)=\frac{du+i\,dv}{dx+i\,dy}, \quad \ldots\ldots\ldots \quad (1)$$

the differentials du and dv are total differentials due to the variation both of x and of y. Thus, by Diff. Calc., Art. 370,

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy, \qquad dv=\frac{dv}{dx}dx+\frac{dv}{dy}dy,$$

in which $\frac{du}{dx}$, $\frac{du}{dy}$ etc. are the partial derivatives of the functions u and v, while dx and dy are independent differentials. Substituting in equation (1),

$$f'(z)=\frac{\left(\frac{du}{dx}+i\frac{dv}{dx}\right)dx+\left(\frac{du}{dy}+i\frac{dv}{dy}\right)dy}{dx+i\,dy}.$$

Since $f'(z)$ has a value independent of the ratio $dy : dx$, we have, by putting dy and dx successively equal to zero,

$$f'(z)=\frac{du}{dx}+i\frac{dv}{dx}=-i\frac{du}{dy}+\frac{dv}{dy}.$$

Equating separately the real and imaginary parts of the last equation, we have the following relations between the partial derivatives:

$$\frac{du}{dx}=\frac{dv}{dy}, \qquad \frac{dv}{dx}=-\frac{du}{dy}. \quad \ldots\ldots \quad (2)$$

Again, eliminating v from these two equations, we have

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0, \quad \ldots\ldots\ldots \quad (3)$$

and the same equation may be found for v.

Two functions of x and y which together satisfy equations (2) are called *conjúgate* functions. Each of them necessarily satisfies equation (3). Thus the functions x^2-y^2 and $2xy$, derived above from the function z^2, are conjugate functions and will be found to satisfy these equations.*

371. From the expressions for u and v in terms of x and y we can readily obtain the equation of the image in the w-plane of a curve in the z-plane whose equation is given. For we have only to eliminate x and y from three given equations. For example, in the case cited above, where

$$u=x^2-y^2, \qquad v=2xy,$$

we thus find, corresponding to $x=a$,

$$v^2=4a^2(a^2-u),$$

which is the equation of a parabola with focus at the origin and vertex on the axis of u to the right of the origin. In like manner, we find corresponding to $y=b$,

$$v^2=4b^2(b^2+u)$$

which is a parabola with focus at the origin and vertex on the axis of u to the left of the origin.

If a series of equidistant values be given to a and also to b, we shall have in the z-plane two sets of straight lines parallel to the axes, dividing the z-plane into small squares. The two corresponding systems of parabolas in the w-plane cut each other

* An expression $u+iv$ in which the functions u and v are taken at random (although a function of z in the sense that it is determined by the position of the point z) is not, unless u and v satisfy the equations above, an *analytical function* of z, that is one which can result from algebraic operations performed upon z, that is, upon $x+iy$ as a whole. Analytical functions were called by Cauchy "fonctions monogènes."

orthogonally, in accordance with Art. 368, and divide the w-plane into curvilinear squares. The correspondence of these small squares in the two planes exhibits most clearly the mode in which the function w varies with the independent variable z.*

Two-valued Functions.

372. Let w be a two-valued function, and let z be represented by the point P moving along a given track from the initial point z_0 to the final point z_1. Then the two corresponding points in the w-plane will in general be distinct points, Q and Q', and will describe tracks, say AB and $A'B'$, which have no common point; that is to say, no point at which Q and Q' arrive simultaneously. Hence, selecting A as the initial value of w, B will in general be determined without ambiguity as the final value of w. But, if the track of z passes through a point for which the two values of w become equal, the paths AB and $A'B'$ will have a common point, and there will be an ambiguity in the final value of w, because it will be possible to pass from A to B'. Such a point in the z-plane is called a *change-point* or *branch-point* for the given function.† Then, provided it does not pass through a branch-point, the track of z (together with the initial value w_0) *determines without ambiguity the final value* w_1.

* Figures illustrating in this way many of the elementary functions will be found in Dr. Thomas S. Fiske's *Functions of a Complex Variable*, p. 13 *et seq.* (Mathematical Monograph Series, John Wiley & Sons, 1906), to which the reader is referred for an excellent introduction to the Theory of Functions.

It will be noticed that, in the case above, illustrating the function z^2, the whole of the w-plane corresponds to one-half of the z-plane. On the other hand, only a portion of the w-plane will correspond to a single value of a multiple-valued function.

† Two values of w may become simultaneously infinite at a branch-point, in which case the conclusions below may be established by means of the function w^{-1}, of which two values become zero.

373. Moreover, if the track of z be altered gradually, that is by continuous or infinitesimal changes, to any other track between z_0 and z_1, the tracks AB and $A'B'$ in the w-plane will suffer continuous change, but it will be impossible to pass from A to B' unless at some instant these tracks have a common point; that is, unless the track of z passes through a branch-point. It follows that a track which can be altered into a given track without ever passing through a branch-point will determine the same final value of w. Such tracks are said to be *reducible*, one to the other. This is as much as to say that, if the area enclosed by the two tracks of z does not contain a branch-point, the tracks lead to the same final value of w; so that, if z returns to z_0 after describing the complete *contour* formed by the two tracks, it will return with the same value of w with which it started. But, if the contour encloses a branch-point, it may return with the other value of w.

374. Take, for example, the simple function $w=\sqrt{z}$. Putting z in the form $re^{i\theta}$, we have seen in Diff. Calc., Art. 228, that w admits of the two values $re^{\frac{1}{2}i\theta}$ and $\sqrt{r}e^{\frac{1}{2}i(\theta+2\pi)}$ which is equal to $-\sqrt{r}e^{\frac{1}{2}i\theta}$. As z moves, the argument θ varies continuously. Now, starting with the initial values $z_0=r_0e^{i\theta_0}$, $w_0=\sqrt{r_0}e^{\frac{1}{2}i\theta_0}$, if z returns to z_0 without encircling the origin, θ will return to θ_0, and w to w_0; but, if z encircles the origin in the positive direction, the final value of the argument of z will be $\theta_0+2\pi$, and the final value of w will be $-w_0$. If z describes the same contour in the negative direction, the final value will again be $-w_0$; but, if it enwraps the origin twice, or any even number of times, w will return to the value w_0.

375. In like manner, if $w=\sqrt{(z-a).\phi(z)}$, a is a branch-point, and a circuit about a described by z will change the sign of w, provided the factor $\phi(z)$ has no branch-point either on or within the circuit described by z. Thus the function

$$w=\sqrt{\{(z-a)(z-b)\}}$$

is two-valued, and has two branch-points, a and b. If z describes a contour enclosing a, but not enclosing b, the sign of w will be changed; and so also if the contour encloses b and not a. But, if the contour encloses both a and b, w will return to the value w_0 when z returns to z_0, so that the function is virtually one-valued for such a contour. Since the direction in which z moves about either of these branch-points is indifferent, the same thing will be true if the track of z forms a figure-of-eight, of which one loop contains a and the other b.

Multiple-valued Functions.

376. The conclusions arrived at in Arts. 372 and 373 apply also to functions having more than two values. Thus, we select an initial value z_0, to this corresponds a number of values of w_0. Then, any track (avoiding branch-points) from the initial point to the point z_1 leads from any one of the values of w_0 to a definite one of the values of w_1. Now, choosing one of the values of w_0 as the initial value, different tracks may lead to different values of w_1; but, if the area between two tracks does not contain a branch-point, the values reached are the same.

A standard track between z_0 and z_1, for example the rectilinear one, will establish a correspondence between the values of w_0 and those of w_1; then one value of w_0 and the corresponding value of w_1 are selected as *primary* values. Now, any track may, without passing through a branch-point, be reduced either to the standard track or to one combining with it one or more *loops* or contours (from and back to the initial point), each surrounding a single one of the branch-points. The result of describing one of these loops is to pass from one to another of the values of w_0, and thus, in connection with the standard track from z_0 to z_1, we may pass from the primary value of w_0 to one of the non-primary values of w_1.

377. For example, let the function be $w = \sqrt[3]{z}$, for which

the origin is the only branch-point. Take unity, represented by the point A in Fig. 66, as the initial value of z, and let B represent the final value of z. Putting $z=re^{i\theta}$, we take zero for the initial value of θ. Then the value of w resulting from a rectilinear track will be $\sqrt[3]{r}e^{\frac{1}{3}i\theta}$, where θ must lie between $-\pi$ and π in value. Call this the primary value of w, so that the primary value of w at A is unity, and at B it is $w_1=\sqrt[3]{r_1}e^{\frac{1}{3}i\theta_1}$.

The result of making the circuit of the origin in the positive direction is to add 2π to the value of θ, and therefore to multiply w by $e^{\frac{2}{3}i\pi}=\omega$, one of the cube roots of unity. (Diff. Calc., Art. 230.) Thus, when the track of z is ACB in Fig. 66, the initial value of w being unity, its final value is ωw_1. Again, the result of making the circuit of the origin in the negative direction is to multiply w by ω^2, the other cube root of unity; so that, when the track of z is ADC, the result is $\omega^2 w_1$. This last is also the result of a track enwrapping the origin twice in the positive direction.

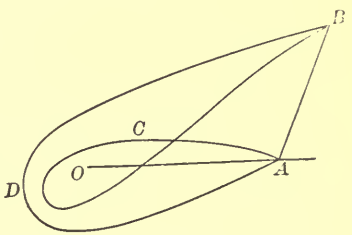

FIG. 66.

378. The function $w=\log z$ presents a branch-point of a different character. We have, on putting

$$z=re^{i\theta}, \qquad w=\log z=\operatorname{Log} r+i\theta;$$

where Log r is the real logarithm of the modulus. Taking unity (with $\theta=0$) for the initial value of z, the initial value of w is zero. Let us denote by the symbol Log z the result when z follows the rectilinear track AB in Fig. 66. Then Log $z=\operatorname{Log} r+i\theta_0$, where θ_0 lies between π and $-\pi$. The circuit of the origin in the positive direction adds 2π to θ, and therefore adds $2i\pi$ to w. Thus, when the track of z is ACB, the final value of w will be Log $z+2i\pi$, and for the track ADB it is Log $z-2i\pi$, while the

general value $\log z = \text{Log } r + i\theta_0 + 2ni\pi$ is the result of n circuits of the origin combined with the rectilinear track AB.

379. The values selected as primary are said to constitute one *branch* of the function, but the mode of selection is arbitrary. A given selection corresponds to a *cut* in the z-plane, or a number of cuts, each starting from a branch-point, and either passing to infinity or terminating in another branch-point. Moreover every branch-point must be one of the extremities of a cut. Starting from any initial point, it is possible to reach any other point of the plane without crossing a cut, and the values of w so obtained constitute the branch. For example, the selection above for the primary value of the logarithm is equivalent to making a cut along the negative part of the real axis.

A *Riemann's Surface* consists of a number of leaves identical with the cut plane and joined to one another along the cut, so that when we cross the cut we pass into another branch of the function. In the case of the logarithm, the number of leaves is infinite, and we ascend, as it were, from leaf to leaf as we go round the branch-point (*Windungspunkt* in German) in one direction. For the function in Art. 375, the cut may be made along the line ab. There are in that case but two leaves, and each is joined to the other along the cut, so that the surface there intersects itself.

A multiple-valued function may be regarded as one-valued on its Riemann's Surface; that is to say, to each point of the surface there corresponds a single value of the function.

Meaning of Integration when the Variable is Complex.

380. Let us now consider the meaning of an integral when z in the expression, $f(z)dz$, under the integral sign admits of complex values. Supposing $F(z)$ to be a function such that its derivative $F'(z) = f(z)$, we write

$$F(z) = \int f(z)dz,$$

in which $f(z)$ is sometimes called *the integrand*, while $F(z)$ is the indefinite integral. To remove the indefiniteness due to the constant of integration we write also, as in Art. 82,

$$\int_a f(z)dz = F(z) - F(a), \quad \ldots \ldots \quad (1)$$

in which a is a selected initial value of z, and it is understood that z in the integral varies continuously from the initial value a to the final value denoted by z in the second member. To the latter, we may of course assign a separate symbol, and place it as the upper limit of the integral.

The initial value a and the final value z in equation (1) may now be complex quantities, and the track described by the current variable z, or *track of integration*, is supposed to be known.

381. Let the indefinite integral be represented by a point $W = F(z)$ in a new plane; then, as z describes the given track, W starting from the position $F(a)$ moves in a direction and with a rate which depend at every instant upon the corresponding value of $f(z)dz$. Provided, therefore, that the track of z does not pass through a point for which $f(z)$ is either infinite or ambiguous * in value, the track of W will terminate in a definite point $F(b)$,

* It will be noticed that the restriction with regard to intermediate points upon the track of z is the same as that of Art. 82 with regard to the intermediate values in the case of real integrals. But, whereas for real integrals this restriction excluded the interpretation of certain integral expressions, no such exclusion now exists. For example, we have when z is a real variable

$$\int_1^z \frac{dz}{z^2} = 1 - \frac{1}{z};$$

but, if the upper limit is negative, the integral is inadmissible, because to reach such a value z must pass through the value zero for which the integrand is infinite. Whereas, when complex values are admitted, the track of integration need not pass through the origin and the equation holds true. Thus, if $z = -1$, the value of the integral is 2.

and, by equation (1), the integral will have a definite value. Thus the *critical points* through which the track of z must not pass include the infinity points, or *infinities*, of the integrand $f(z)$ as well as its branch-points.

These critical points are the only positions at which branch-points of the indefinite integral $F(x)$ can occur. Hence it follows, as in Art. 373, that any two tracks which can be reduced one to the other without passing through a critical point will give the same value to the integral.

Integration around a Closed Contour.

382. It follows from the preceding article that, when z describes a closed curve or contour returning to its initial value, the integral must vanish unless the contour encloses one or more critical points. Furthermore, in comparing the results of different tracks of integration between the same limiting points, we may adopt, as in Art. 376, a standard track, and reduce any other track to a combination of this with one or more *loops* or contours, each enclosing a single critical point. For example, let us take the integral of dz/z, for which the origin is an infinity of the integrand. Let

$$I_0 = \int_c^c \frac{dz}{z}$$

denote the result of integrating from an initial point c in a contour described in the positive direction about the origin and back to c. Putting $c = re^{i\theta_0}$, we may take for the contour a circle of radius r and centre at the origin. This is equivalent to putting $z = re^{i\theta}$ and making r constant while θ varies from θ_0 to $\theta_0 + 2\pi$. We have now $dz = ire^{i\theta}d\theta$, hence

$$I_0 = i\int_{\theta_0}^{\theta_0 + 2\pi} d\theta = 2i\pi.$$

Thus it appears that, in this case, the value of the contour integral is independent of the initial point c. Taking unity as the initial point, the value of the indefinite integral is $\log z$, and $2i\pi$ is the difference between consecutive values of this many-valued function. Compare Art. 378.

Integration about a Pole.

383. If a is an infinity of $f(z)$, we shall assume that there is a positive value of n such that the product $(z-a)^n f(z)$ has a finite value when $z=a$. Denoting this product by $\phi(z)$, we have

$$f(z)=\frac{\phi(z)}{(z-a)^n},$$

where $\phi(z)$ is such that $\phi(a)$ is neither zero nor infinite. If n is an integer, a is called a *pole* of $f(z)$ of the order n. For example, $\tan z$ has a pole of the first order, or simple pole, at $\frac{1}{2}\pi$; because $(z-\frac{1}{2}\pi)\tan z$ has a finite value when $z=\frac{1}{2}\pi$. If n is fractional, a is a branch-point as well as an infinity of $f(z)$.

We shall now show that *the result of integrating $f(z)$ in a contour about a pole, not enclosing any other critical point, is independent of the initial point.* In Fig. 67, let a be the pole, c the initial point of a contour cdc and b any other point from which a contour beb is described about a in the same direction. Since by hypothesis there is no critical point between the contours, the contour cdc may be reduced without passing through a critical point to $cbebc$, consisting of the straight line cb followed by the contour beb about a returning to b, and then back to c by the straight line. Since $f(z)$ has no branch-point within the contour, it must return to b with the same value with which it commences to describe the loop beb,

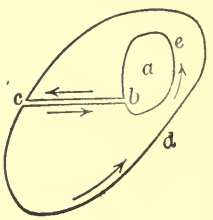

FIG. 67.

so that the two rectilinear portions of the integral cancel each other. Hence the two contour integrals must be equal.* In other words, the contour integral or *loop-integral* is *independent of the initial value.* Its value may be denoted by I_a to indicate its dependence upon the pole a.

384. To obtain the value of I_a in the case of a simple pole, that is, when $n=1$, we notice that if $\phi(z)$ were constant, we should have, by putting $z-a=re^{i\theta}$ as in Art. 382, the result $I_a=2i\pi\phi(a)$. But, by the preceding article, this must also be the result when $\phi(z)$ is variable, because we can give the circular contour in the demonstration as small a radius as we please.

The result may also be established as follows: The function $\dfrac{\phi(z)-\phi a}{z-a}$ has a finite value, namely $\phi'(z)$, when $z=a$, therefore it has no critical point within the contour. Hence, by Art. 382,

$$\int_c \frac{\phi(z)-\phi(a)}{z-a}dz=0;$$

therefore

$$I_a=\int_C \frac{\phi(z)dz}{z-a}=\phi(a)\int_C \frac{dz}{z-a}=2\pi i.\phi(a), \quad . \quad . \quad . \quad (1)$$

where the suffix C indicates a contour of integration about a.

385. When the pole is of the second order, the result of putting $z-a=re^{i\theta}$ and making r constant is

$$I_a=\int_c \frac{\phi(z)\,dz}{(z-a)^2}=\frac{i}{r}\int_0^{2\pi}\frac{\phi(a+re^{i\theta})}{e^{i\theta}}d\theta.$$

This expression, in which r can be diminished indefinitely, takes the indeterminate form when $r=0$ (because the integral of $e^{-i\theta}d\theta$ between 0 and 2π vanishes), and evaluating by the usual process

* The rectilinear portions of the reduced track are separated by a small space in the figure to show that equivalent contours must be described in the same direction.

we have $I_a = 2i\pi\,\phi'(a)$. This result is also deducible from equation (1) above by taking derivatives with respect to a; and, in like manner, by taking successive derivatives we have the series of equations:

$$\int_C \frac{\phi(z)dz}{z-a} = 2i\pi\phi(a), \quad \ldots\ldots (1)$$

$$\int_C \frac{\phi(z)dz}{(z-a)^2} = 2i\pi\phi'(a), \quad \ldots\ldots (2)$$

$$2\int_C \frac{\phi(z)dz}{(z-a)^3} = 2i\pi\phi''(a), \quad \ldots\ldots (3)$$

$$2.3\int_C \frac{\phi(z)dz}{(z-a)^4} = 2i\pi\phi'''(a), \quad \ldots\ldots (4)$$

and, in general,

$$n!\int_C \frac{\phi(z)dz}{(z-a)^{n+1}} = 2i\pi\phi^{(n)}(a). \quad \ldots\ldots (5)$$

The contours are here supposed to be described in the positive direction, and, if the pole is enwrapped k times, the result is multiplied by k.

Again, if a function has two poles a and b, it is readily seen that the result of a contour encircling both a and b will be $I_a + I_b$.

Integrals of Functions with Poles.

386. It follows from the preceding articles that the integral of a function with poles will be a function having an unlimited number of values differing by multiples of one or more constants dependent upon the poles.

Take, for example, the integral

$$\int_0 \frac{dz}{z^2+1} = \int_0 \frac{dz}{(z-i)(z+i)}.$$

The poles of the integrand are the points i and $-i$. The contour integral about the first is, by equation (1),

$$I_i = 2i\pi \frac{1}{z+i}\Big]_{z=i} = \pi;$$

and, in like manner, we find $I_{-i} = -\pi$.

The initial point is so taken that, for real positive or negative values of z, the integral is *the primary value* of $\tan^{-1}z$, as defined in Diff. Calc., Art. 76. The rectilinear track of integration may now be used to define the primary value of the function for all points except those on the imaginary axis beyond the points i and $-i$. This selection of the primary value therefore corresponds to cuts of the plane (see Art. 379) made along these portions of the axis. It follows that any circuit which crosses a cut from the right side to the left of the axis adds π to the value of the function, and one crossing in the reverse direction subtracts π. In particular, a circuit about both poles restores the value of the function.

Art. 382 and the present one illustrate the fact that, when the integrand has a pole, integration gives rise to a function which admits of an infinite number of values differing by multiples of a constant. It follows that the inverse of such a function is a *periodic* function, in which values of the independent variable differing by multiples of a constant (called the period) correspond to the same value of the function. Thus e^z and $\tan z$ are periodic functions, the first having the pure imaginary period $2i\pi$ and the second the real period π.

Integration about a Branch-Point.

387. The value of a contour integral about a branch-point of the integrand, unlike that about a pole, depends upon the position of the initial point. The reason is that when a circuit of the

branch-point is completed the integrand returns with a new value; thus, referring to Fig. 67, if a were a branch-point, the integrand in the contour *cbebc* would have a different value when z finally leaves b from that with which it first arrived at b, hence the two rectilinear parts of the integral would no longer cancel one another.

In illustration, let us integrate $z^n\,dz$ in a circle whose centre is the origin, from the initial point c back to c. Except when n is an integer, the origin is a branch-point of the integrand. Putting $c=re^{i\theta_0}$ and $z=re^{i\theta}$, the contour is described by making r constant while θ varies from θ_0 to $\theta_0+2\pi$. Hence the contour integral is

$$\int_c^c z^n\,dz = ir^{n+1}\int_{\theta_0}^{\theta_0+2\pi} e^{i(n+1)\theta}\,d\theta = \frac{r^{n+1}}{n+1}\,e^{i(n+1)\theta}\Big]_{\theta_0}^{\theta_0+2\pi}$$

$$= \frac{r^{n+1}}{n+1}\,e^{i(n+1)\theta_0}\left[e^{2i(n+1)\pi}-1\right] = \frac{c^{n+1}}{n+1}\left[e^{2i(n+1)\pi}-1\right].$$

Now $\dfrac{z^{n+1}}{n+1}$ is the indefinite integral, and $e^{2i(n+1)\pi}$ is one of the complex values of 1^{n+1}, hence the contour integral is (as we should expect) the quantity which must be added to one of the multiple values of the integral in order to produce another.

In this example, if n is zero or a positive integer, the contour integral vanishes, the indefinite integral being in that case a one-valued function. When $n=-1$, the contour integral takes the indeterminate form, and is found on evaluation to be $2i\pi$, agreeing with Art. 382, and when n is any other negative integer, the contour integral vanishes in accordance with equation (5), Art. 385.

Integrals involving Radicals.

388. When an integral involves an ordinary quadratic radical, every point at which the quantity under the radical sign vanishes is a branch-point. If the branch-point is on the real

axis, and a real initial value is chosen, the value of the contour integral is determined by a real definite integral.

Take for example the integral

$$V=\int_0 \frac{dz}{\sqrt{(1-z^2)}}.$$

The critical points are $+1$ and -1. The contour from O about A, the first of these points, may be reduced, as in Fig. 68, to the rectilinear track from O to a point as near as we choose to A, followed by a circular track about A, and t e return to O by the rectilinear track. The effect of the circular part is to change the sign of the radical, so that we may write

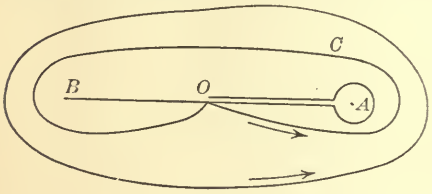

FIG. 68.

$$I_A=\int_0^{1-\delta} \frac{dz}{\sqrt{(1-z^2)}}+\text{result of circular track}+\int_{1-\delta}^{0} \frac{dz}{-\sqrt{(1-z^2)}},$$

in which the results of the two rectilinear parts are equal. For the circular part, put $z=1-\delta e^{i\theta}$, so that the circle is described by keeping δ constant and varying θ from 0 to 2π. Then $dz=-i\delta e^{i\theta}$, and the equation becomes

$$I_A=2\int_0^{1-\delta} \frac{dz}{\sqrt{(1-z^2)}}-i\sqrt{\delta}\int_0^{2\pi} \frac{e^{\frac{1}{2}i\theta}d\theta}{\sqrt{(2-\delta e^{i\theta})}},$$

in which δ may be decreased without limit The second term vanishes with δ; hence, putting $\delta=0$, we have

$$I_A=2\int_0^{1} \frac{dz}{\sqrt{(1-z^2)}}=\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

389. The direction in which the circuit of A is made does not affect the value of I_A, but it must be remembered that the

value given in equation (1) implies that the initial point is the origin and that the initial value of the radical is $+1$. Since the sign of the radical is changed by the circuit, the result of a second circuit would be $-\pi$. That is, the result of two circuits about A would be zero, and the integrand would return to its original value.

In like manner, we shall find $I_B = -\pi$, assuming as before that the initial value of the radical is $+1$. But, if the circuit of B is made after the circuit of A (which changes the sign of the radical), the result will be π, and the whole result of a contour encircling first A and then B, like OCO in Fig. 68, will be 2π. In this contour the integrand is virtually one-valued, see Art. 375, whence it can be shown, exactly as in Art. 383, that the value of the contour integral is independent of the initial point. Hence, for the contour which does not pass through O, as well as for OCO, we may write $I_{AB} = 2\pi$. The two branch-points are thus together equivalent to a pole.*

390. Let us now, for any point z, define the primary value V_1 of the integral V as the result of integrating along the rectilinear track Oz in Fig. 69. Consider now the result of integrating along the track OCz. This track may be reduced to the contour about A followed by the track Oz. Hence, remembering that the sign of the integrand is changed by the contour, the result is $\pi - V_1$.

The results obtained in the preceding article also show that a contour encircling A twice before reaching z will produce V_1.

* Accordingly, if z describes a very large circle with centre at the origin, the integral approaches the form $\int \frac{dz}{\sqrt{(-z^2)}} = \pm i \int \frac{dz}{z}$, for which, by Art. 382, the loop-integral is $\pm 2\pi$. The ambiguous sign is due to the fact that we have not determined which of its two values to give the integrand at any one point of the circuit. In Fig. 68 we obtained $+2\pi$ for the circuit OCO, because we assumed the radical to have the value $+1$ at O.

One encircling B, like ODz in the figure, produces $-\pi - V_1$. Again, one encircling A and then B in either direction produces $2\pi + V_1$. Thus, by means of different tracks, we can reach any of the values included in the formulæ

$$2n\pi + V_1 \qquad \text{and} \qquad (2n+1)\pi - V_1,$$

where n is an integer.

The integral defines the function $\sin^{-1}z$ for complex values of z, and we have thus shown that the relations which exist between the multiple values when z is real hold also when z is complex.

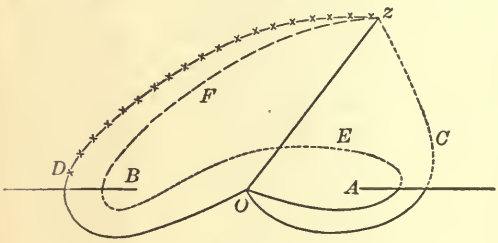

FIG. 69.

The fact that a circuit of both branch-points is equivalent to that of a pole gives a periodic character to the inverse function, just as in Art. 386.

391. The use of the rectilinear track to define the primary branch of this function is equivalent to making cuts in those parts of the real axis which are beyond the points A and B. The Riemann surface consists of an infinite number of leaves. If L_0, L_1, L_2, L_3 etc. denote consecutive leaves, L_0 and L_1 may be joined along the positive cut so as to form a self-intersecting surface (as in the second illustration used in Art. 379), and L_2 and L_3 are joined in like manner. But, along the negative cut L_1 and L_2 must be thus joined, and L_3 in like manner with L_4, and so on.

In Fig. 69, the parts of the several tracks which would lie on different leaves of the surface thus constructed are indicated. The initial point O being on L_0, we arrive at z by the route

through C on the leaf L_1; by the route through E and F, we arrive on the leaf L_2; and, by the route through D, we arrive on the leaf L_{-1}.

The Modulus of a Sum.

392. The sum of several complex quantities $a, b, \ldots, l$ is graphically represented by the straight line OL, where $OA, AB, \ldots, KL$ (equal to the moduli of the given quantities) are laid off successively each in its proper direction, forming a broken line beginning at the origin. Therefore *the modulus of the sum*, which is the length of OL, *cannot exceed the sum of the moduli*, which is the length of the broken line $OBC \ldots L$. It is in fact equal to this sum only when the arguments of all the terms are the same, so that the broken line becomes a straight line.

Applications of this principle in the discussion of infinite series occur in the following articles. We here notice its application to the value of a definite complex integral. Let

$$w = \int_L f(z)dz,$$

where $\int_L$ denotes a specified track of integration in the z-plane. While z describes the track L, w describes a track in the w-plane. Regarding the integral as the sum of its elements, this last track is the limiting form of the broken line representing the summation of the elements, hence its length is the limit of the sum of the moduli of the elements. If s is the length of arc in the path of z, ds is the modulus of dz, and if μ is the variable modulus of $f(z)$, $\mu\, ds$ is the modulus of an element; thus $\int_L \mu\, ds$ is the length of the track in the w-plane. The modulus of w therefore cannot exceed this integral. Let M be the greatest value of μ for points

on the track L, and let l be the length of that track; it follows *a fortiori* that the modulus of w cannot exceed Ml.

Power Series in the Complex Variable.

393. A series of complex terms is *convergent* when the point representing the sum of n terms tends to a fixed limiting position as n increases without limit. Among non-convergent series, we may distinguish between *oscillating* series for which this point remains at a finite distance (but does not approach a limiting value), and properly *divergent* series for which the point recedes indefinitely from the origin.

A series is necessarily convergent when the moduli of its terms form a convergent series (the terms of which are of course all positive). Such a series is said to be *absolutely convergent.* For example the series

$$1+z+z^2+\ldots$$

is absolutely convergent when z is a complex quantity whose modulus is less than unity. In Fig. 70 the sum of a number of terms of this series is constructed. The broken line $OABC\ldots$ here consists of links whose lengths form a decreasing geometrical progression, and their inclinations to the axis of reals an arithmetical progression, so that the angles at A, B, C, etc., are all equal. Such a polygon can be inscribed in an equiangular spiral,* hence the pole P of this spiral is the

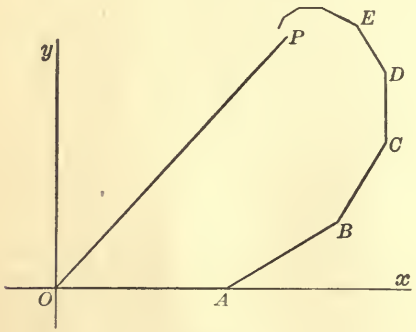

FIG. 70.

* See Diff. Calc., Art. 324. If successive radii-vectores of an equianguiar spiral make equal angles at the pole, they are in geometrical progression, so

limiting position of the nth vertex when n increases without limit, and OP represents the sum of the series.

394. When the modulus of z is greater than unity, the nth vertex of the broken line is on an equiangular spiral but recedes from the pole indefinitely. In the intermediate case, when the modulus is unity, the polygon is plainly one inscribed in a circle. We can only say of the nth vertex that it must lie somewhere on this circle; therefore the sum of n terms does not tend toward a limit, but oscillates about a mean value.

An example of a convergent series which is not absolutely convergent is furnished by the series

$$z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\cdots+\frac{1}{n}z^n+\cdots,$$

when the modulus of z is unity. The series of moduli is the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots$, which we have seen has an infinite sum (Diff. Calc., Art. 180). But, when z is a complex unit, the series is convergent; the broken line, though of infinite length, is wrapped about a limiting point. So also, when $z=-1$, we have a series of alternately positive and negative terms, and the sum converges to the limit log 2.*

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that the triangles formed by joining their extremities are similar. Hence the chords are in geometrical progression and they make equal angles with one another.

* The peculiarity of the absolutely convergent series is that there is but one limiting value, independent of the order of the terms, whereas in the other case any limiting values or even an infinite value may be reached as the result of a different law of succession of the terms. For example, in the series of real terms of alternating signs considered above, the sum of the terms of either sign is infinite, and it is only when taken alternately that their sum approaches the limit log 2. A series of this kind is said to be *semi-convergent*. An analogous case in infinite products is considered in Art. 344, see foot-note.

Circle and Radius of Convergence.

395. If the power series

$$A+Bz+Cz^2+Dz^3+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is convergent for a given value Z, it will be convergent for any value of z whose modulus is less than that of Z.

Let R be the modulus of Z and $r<R$ the modulus of z, then

$$1+\frac{r}{R}+\frac{r^2}{R^2}+\frac{r^3}{R^3}+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is a convergent geometrical series whose sum is $R/R-r$. Now, since the series

$$A+BZ+CZ^2+DZ^3+\cdots \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is by hypothesis convergent, its terms are all finite. Let M denote the greatest modulus of any term of the series (3). The terms of the series (2) are the ratios of the moduli of corresponding terms of (1) and (3); hence

$$M\left(1+\frac{r}{R}+\frac{r^2}{R^2}+\frac{r^3}{R^3}+\cdots\right)=\frac{MR}{R-r}$$

exceeds the sum of the moduli of the series (1). It follows that these moduli form a convergent series, and the series (1) is absolutely convergent when $r<R$.

396. A power series may be convergent for all values of z. If it is not, let R be *the greatest modulus* of any point for which

it is convergent, then it will be convergent for every point whose modulus is less than R, that is to say, for every point within the circle whose radius is R and centre at the origin. Again, it will be divergent for every point outside of this circle, because R is by hypothesis the greatest modulus for which it is convergent. This circle is called *the circle of convergence* for the series, and R is called *the radius of convergence.*

It is to be noticed that nothing is proved as to the points on the circumference of the circle. The extremities of the diameter lying in the axis of reals are the "limits of convergence" for power series of the real variable, and we have seen that such a series may be convergent at one limit and divergent at the other (Diff. Calc., Art. 182).

Taylor's Series.

397. The equations of Art. 385 may be written in the form

$$\phi(a)=\frac{1}{2i\pi}\int_C \frac{\phi(z)dz}{z-a},$$

$$\phi'(a)=\frac{1}{2i\pi}\int_C \frac{\phi(z)dz}{(z-a)^2},$$

. . . . . .

$$\phi^{(n)}(a)=\frac{n!}{2i\pi}\int_C \frac{\phi(z)dz}{(z-a)^{n+1}},$$

no critical point of the function $\phi(z)$ being either on or within the contour of integration C.

These equations are the starting-point for many investigations in the theory of functions. They enable us, for example, to prove Taylor's Theorem for the complex variable and to determine the radius of convergence as follows:

398. Let the contour C be a circle with radius r about the point a as centre, Fig. 71, and let k, the modulus of h, be less than r, so that the point $a+h$ is within the circle. Since the circle is a contour about $a+h$ and there is no critical point of $\phi(z)$ either on or within its circumference,

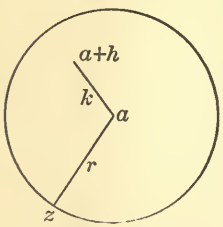

FIG. 71.

$$\phi(a+h)=\frac{1}{2i\pi}\int_C \frac{\phi(z)dz}{z-a-h}. \quad . \quad . \quad (1)$$

Now, n being any positive integer,

$$\frac{1}{z-a-h}=\frac{1}{z-a}+\frac{h}{(z-a)^2}+\cdots+\frac{h^n}{(z-a)^{n+1}}+\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}.$$

Substituting in equation (1),

$$\phi(a+h)=\frac{1}{2i\pi}\left[\int_C \frac{\phi(z)dz}{z-a}+h\int_C \frac{\phi(z)dz}{(z-a)^2}+\cdots+h^n\int_C \frac{\phi(z)dz}{(z-a)^{n+1}}+\right]R_n,$$

where

$$R_n=\frac{h^{n+1}}{2i\pi}\int_C \frac{\phi(z)dz}{(z-a)^{n+1}(z-a-h)}.$$

Hence

$$\phi(a+h)=\phi(a)+h\phi'(a)+\frac{h^2}{2!}\phi''(a)+\cdots+\frac{h^n}{n!}\phi^{(n)}(a)+R_n. \quad (2)$$

We have now to show that the remainder R_n has zero for its limit when n increases without limit. For this purpose, consider the modulus of R_n. The modulus of $z-a$ remains constant and equal to r while z moves around the contour. We have, therefore,

$$\text{mod. } R_n=\frac{k^{n+1}}{2\pi r^{n+1}}\,\text{mod.}\int_C \frac{\phi(z)dz}{z-a+h}.$$

The integral in this expression is finite by Art. 392 because $\phi(z)$ is by hypothesis finite for every point of the contour and $z-a+h$ does not vanish. Hence, because the fraction k/r is less than unity, the modulus of R_n decreases without limit as n increases, that is to say, the series is absolutely convergent.

Let R be the distance from a of the nearest critical point of $\phi(z)$: then r may be any quantity less than R and as near to it as we please. Therefore R is the radius of convergence.

399. The result of putting $a=0$, and writing z in place of h, is of course Maclaurin's series

$$f(z)=f(0)+f'(0)z+f''(0)\frac{z^2}{2!}+\cdots+f^{(n)}(0)\frac{z^n}{n!}+\cdots,$$

in which $f(z)$ is developed into a series involving positive integral powers of z. Hence, if the origin is not itself a critical point, the function can be developed in powers of z within a circle whose radius R is the least modulus of a critical point.

For example, in the case of the function $\tan^{-1}z$, its integral form, Art. 386, shows that the limits of convergence for the corresponding series are ± 1, because unity is the modulus of each of the poles i and $-i$.

A one-valued function which, like e^z, is infinite only for infinite values of z can be developed into a series convergent for the whole plane.

400. We can now show that a one-valued function cannot fail to become infinite, for at least one value of z, finite or infinite. For, if this were possible, the modulus of z would have some finite maximum value M. Now we have, for auy point a,

$$\phi^{(n)}(a)=\frac{n!}{2i\pi}\int_C\frac{\phi(z)dz}{(z-a)^{n+1}},$$

in which, since $\phi(z)$ has by hypothesis neither pole nor branch-point, the contour C may be taken as a circle with centre a and any radius. Putting, then, $z=a+re^{i\theta}$, whence $dz=ire^{i\theta}d\theta$, we have

$$\phi^{(n)}(a)=\frac{n!}{2i\pi}\int_0^{2\pi}\frac{\phi(z)ire^{i\theta}\,d\theta}{r^{n+1}e^{(n+1)\theta}}=\frac{n!}{2\pi r^n}\int_0^{2\pi}\phi(z)e^{-ni\theta}d\theta.$$

The modulus of the integral in the last expression is, by Art. 392, less than $2\pi M$; hence the modulus of $\phi^{(n)}(a)$ cannot exceed $n!M/r^n$. In this expression r can be made as large as we choose, hence $\phi'(a)$ $\phi''(a)$ etc., all vanish, and equation (2), Art. 398, becomes $\phi(a+h)=\phi(a)$; that is to say, $\phi(z)$ reduces to a constant and is not a function of z.

Since this applies also to the function $1/\phi(z)$, it follows that $\phi(z)$ must become zero, either when z is infinite or for some finite value of z. In particular, if $\phi(z)$ is a rational integral function we have thus proved the fundamental theorem of Algebra that $\phi(z)=0$ must have a root.

Examples XXIII.

1. Putting $w=\rho e^{i\psi}=z^2$, derive the polar equations of the images of lines parallel to the axes of x and of y. *Compare Art.* 371.

$$\rho=\frac{2a^2}{1+\cos\psi};\quad \rho=\frac{2b^2}{1-\cos\psi}.$$

2. Derive conjugate functions from $w=z^3$, and thence the equations of the images of $x=a$ and $y=b$.

$$u=x^3-3xy^2,\qquad 27a^3v^2=(a^3-u)(8a^3+u)^2,$$
$$v=3x^2y-y^3;\qquad 27b^3u^2=(b^3+v)(8b^3-v)^2.$$

3. Derive conjugate functions from $w=e^z$, and show that the net-work of lines parallel to the axes in the z-plane is transformed into concentric circles and their radii, while the images of all other straight lines are equiangular spirals.

$$e^x\cos y,\quad e^x\sin y.$$

4. Show that $u=a$ and $v=b$ give for the inverse of a given function the systems of curves corresponding to parallels to the axes. Thence show that, for the function $z=\log w$, these systems are

$$x=\log \sec y+C, \qquad x=\log \operatorname{cosec} y+C'.$$

Verify that these curves cut orthogonally.

5. Show that, for the function square-root, both systems of curves are rectangular hyperbolas; and, for the cube-root, each consists of cubics having three fixed asymptotes.

6. Show that the polar equations of the images of $x=a$ and $y=b$ in Ex. 2 are

$$\rho=a^3 \sec^3 \tfrac{1}{3}\psi, \qquad \rho=b^3 \operatorname{cosec}^3 \tfrac{1}{3}\psi.$$

Trace these curves for $a=1$ and $b=1$ and verify their orthogonal intersection at $(-2, 2)$. Show also that, at the other two points of intersection, these curves must cut each other at angles of 30°.

Notice that the image of $x=1$ is also the image of other straight lines in the z-plane.

7. Given $w=z^n$, show that lines parallel to the axes are converted into circles when $n=-1$; into cardioids when $n=2$; and into lemniscates when $n=-\frac{1}{2}$.

8. If $w=\sqrt{z}$, show that, when z describes a small circle about a given point a, the two branches of the function w will describe the separate branches of a Cassinian (Diff. Calc., p. 313, Ex. 29); if the circle passes through the origin, the ovals merge into a lemniscate; and if it encloses the origin, the two branches of the function describe halves of a continuous oval.

9. Given $w=\sqrt[3]{z}$, assuming the initial values $z_0=1$, $w_0=1$, draw a track of z which will make w the real cube-root of a negative quantity.

10. Putting $a=1$, $b=-1$ in the example given in Art. 375, so that

$$w=\sqrt{(z^2-1)},$$

show that the systems of hyperbolas

$$x^2-y^2=A \qquad \text{and} \qquad xy=B$$

form an orthogonal net-work, of which the images are

$$u^2-v^2=A-1 \qquad \text{and} \qquad uv=B,$$

in the w-plane. Show by construction that arcs of the hyperbolas $x^2-y^2=\frac{1}{2}$, $xy=-\frac{3}{16}$, $x^2-y^2=\frac{9}{4}$, $xy=\frac{3}{16}$ form a contour by which z can pass from the value $z_0=\frac{3}{4}+\frac{1}{4}i$, around the branch-point $z=1$, back to z_0. Then, by tracing the images of these hyperbolas, show that w, starting from the value $w_0=\frac{1}{4}+\frac{3}{4}i$, will, while z completes the contour, pass from w_0 to $-w$ in accordance with Art. 375.

11. If $w=\sin z$, we have

$$u=\sin x \cosh y, \qquad v=\cos x \sinh y$$

(Diff. Calc., Art. 220); show that the orthogonal net-work in the w-plane corresponding to lines parallel to the axes in the z-plane consists of confocal ellipses and hyperbolas.

Note that for the inverse function, $z=\sin^{-1} w$, the foci are the branch-points, and the ellipse furnishes an example of the contour about both branch-points equivalent to the circuit of a pole. See Art. 389.

12. Given $w=\log\dfrac{z-a}{z-b}$, show that lines parallel to the axes in the w-plane correspond to two systems of circles in the z-plane. If the z-plane be cut by a circular arc joining a and b, the cut plane corresponds to a horizontal strip of the w-plane.

Put $z-a=\rho_1 e^{i\theta_1}$, $z-b=\rho_2 e^{i\theta_2}$.

13. Putting $a=-i$ and $b=i$ in example 12, show that $w=2i\cot^{-1}z$, whence $z=\cot\dfrac{w}{2i}$. Hence show how to construct two circular arcs in the z-plane whose intersection determines $z=\cot(c+id)$.

A and B being the points $-i$ and i, one passes through the points where AB is cut harmonically in the ratio e^{-2d}; the other is a segment on AB containing the angle $2c$.

14. Prove the result mentioned in the foot-note on p. 415 by integration along a semicircular track.

15. Show geometrically that, when z describes a circle of radius a about the origin, the value of dz/z is ids/a; and thence derive the value of the loop-integral.

16. Find the value of $\int_C \frac{\phi(z)}{(z-a)(z-b)}dz$ when the contour C includes a and b but no critical point of $\phi(z)$; and thence deduce equation (2), Art. 385.

17. Find the values of the loop-integrals of dz/z^2+i.

$$\pm 2\sqrt{2\pi(1+i)}.$$

18. Find the values of the loop-integrals of dz/z^3-1.

$$\frac{2}{3}\pi i; \quad -\frac{\pi}{3}(i\pm\sqrt{3}).$$

19. Given $\int\frac{dz}{1+z^4}$, find the values of the loop-integrals for the poles in the first and second quadrants, and show that the result of integrating in the positive direction about these two poles is equivalent to the rectilinear integral from $-\infty$ to $+\infty$.

$$I_A=\frac{\pi(1-i)}{2\sqrt{2}}.$$

Compare Ex. XX, 2.

20. Given $\int\frac{dz}{(1+z^3)\sqrt{z}}$, find the result of integration in a contour including the three poles but not the branch-point, assuming the argument of z to lie always between 0 and 2π. $\frac{4}{3}\pi$.

21. Construct the point representing the generating function of the series represented by Fig. 70, and show that it is the pole of the equiangular spiral mentioned in Art. 393. Show also that, when the modulus of z is unity, it is the centre of the circle mentioned in Art. 394.

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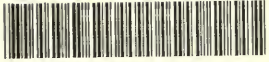
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